Lie Algebras

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I Lie Algebras

Definition I.1. Let L be a vector space over a field F, and $[\cdot,\cdot]\colon L \times L \to L$ be a bilinear map satisfying:

(L1) $\forall x \in L$, $[x,x] = 0$

(L2) $\forall x,y,z \in L$, $[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$

Then $(L,[\cdot,\cdot])$ is a *Lie algebra*, and $[\cdot,\cdot]$ is a *Lie bracket*.

(L2) is called the Jacobi identity.

Lemma I.2. *Let $(L,[\cdot,\cdot])$ be a Lie algebra, then:*

(L1*) $\forall x,y \in L$, $[x,y] = -[y,x]$.

Proof.

$$\begin{aligned} 0 &= [x+y,x+y] = [x,x] + [x,y] + [y,x] + [y,y] \\ &= [x,y] + [y,x] \end{aligned}$$

■

Exercise. Let L be a v.s. over a field F of characteristic not equal to 2. Suppose that $[\cdot,\cdot]\colon L \times L \to L$ is bilinear and satifies (L2) and (L1*), then $(L,[\cdot,\cdot])$ is a Lie algebra.

Examples.

1. Let V be a v.s. over F. Define $[\cdot,\cdot]\colon V \times V \to V$ by $[v,w] = 0$. Then $(V,[\cdot,\cdot])$ is a Lie algebra. This is called an abelian Lie algebra.

2. Let $F = \mathbb{R}$, $L = \mathbb{R}^3$. Define $[\cdot,\cdot] = \cdot \times \cdot$. Then $(\mathbb{R}^3,[\cdot,\cdot])$ is a Lie algebra.

3. Let F be a field, $M_n(F) := \{A \mid A \text{ is an } n \text{ by } n \text{ matrix over } F\}$. Define $[\cdot,\cdot]\colon M_n(F) \times M_n(F) \to M_n(F)$ by $[A,B] = AB - BA$. Then $(M_n(F),[\cdot,\cdot])$ is a Lie algebra denoted $\mathfrak{gl}(n,F)$.

3′. Let V be a v.s. over F of dimension n, $\operatorname{End} V := \{T\colon V \to V \mid T \text{ is linear}\}$. Define $[\cdot,\cdot]\colon \operatorname{End} V \times \operatorname{End} V \to \operatorname{End} V$ by $[T_1,T_2] = T_1 \circ T_2 - T_2 \circ T_1$. Then $(\operatorname{End} V,[\cdot,\cdot])$ is a Lie algebra, $\mathfrak{gl}(V)$.
Question: Is there a connection between 3 and 3′?

4. Recall that for $A \in M_n(F)$, $\operatorname{tr} A := \sum_{i=1}^{n} a_{i,i}$.
$\mathfrak{sl}(n,F) := \{A \in M_n(F) \mid \operatorname{tr} A = 0\}$ with $[A,B] = AB - BA$ is a Lie algebra.

5. As is $\mathfrak{b}(n,F) := \{A \in M_n(F) \mid a_{ij} = 0 \text{ for } i > j\}$ with $[A,B] = AB - BA$.

6. And $\mathfrak{u}(n,F) := \{A \in M_n(F) \mid a_{ij} = 0 \text{ for } i \geqslant j\}$ with $[A,B] = AB - BA$.

7. $\left\{ \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right) \in M_{2n}(F) \mid A,B,C,D \in M_n(F),\ B^T = B,\ C^T = C,\ A^T = -D \right\}$
with $[M,\widetilde{M}] = M\widetilde{M} - \widetilde{M}M$ is a Lie algebra denoted $\mathfrak{sp}(2n,F)$.

8. $\{A \in M_{2n+1}(F) \mid SA = -A^T S\}$ where $S = \left(\begin{array}{c|cc} 1 & \multicolumn{2}{c}{0} \\ \hline 0 & 0 & I_n \\ & I_n & 0 \end{array}\right)$. This forms a Lie algebra with $[A,B] = AB - BA$ denoted $\mathfrak{o}(2n+1, F)$.

Exercise. Check that $\forall A \in \mathfrak{o}(2n+1, F)$, $A = \left(\begin{array}{c|cc} 0 & b & c \\ \hline d & B & C \\ e & D & E \end{array}\right)$ where $d = -c^T$, $e = -b^T$, $C^T = -C$, $D^T = -D$, $B^T = -E$.

The dimension of a Lie algebra is its dimension as a vector space. In this course all Lie algebras will be finite dimensional.

Consider $\mathfrak{gl}(n, F)$ with $n = 2$.

$$A \in \mathfrak{gl}(2, F) \Rightarrow A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ae_{11} + be_{12} + ce_{21} + de_{22}$$

$\dim \mathfrak{gl}(2, F) = 4$

$$\begin{aligned} [e_{ij}, e_{ij}] &= 0 & & \\ [e_{11}, e_{22}] &= 0 & [e_{12}, e_{21}] &= e_{11} - e_{22} \\ [e_{11}, e_{12}] &= e_{12} & [e_{22}, e_{12}] &= -e_{12} \\ [e_{11}, e_{21}] &= -e_{21} & [e_{22}, e_{21}] &= e_{21} \end{aligned}$$

Now if $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$, then

$$\begin{aligned} [A,B] &= [a_{11}e_{11} + \cdots, b_{11}e_{11} + \cdots] \\ &= a_{11}b_{11}[e_{11}, e_{11}] + a_{11}b_{12}[e_{11}, e_{12}] + \cdots \\ &= (a_{11}b_{12} - a_{12}b_{11})e_{12} + \cdots \end{aligned}$$

So this is enough to determine the Lie bracket for all pairs of matrices.

To generalise this, let L be a Lie algebra with a basis $(e1, \ldots, e_n)$ and suppose we know $[e_i, e_j] = \sum_{k=1}^n c_{ij}^k e_k$ for all i, j where $c_{ij}^k \in F$. Take $A, B \in L$, $A = \sum_{i=1}^n a_i e_i$, $A = \sum_{j=1}^n b_i e_i$ with $a_i, b_i \in F$. Then $[A, B] = [\sum a_i e_i, \sum b_i e_i] = \sum_{i<j} (a_i b_j - a_j b_i)[e_i, e_j]$. So to determine $[\cdot, \cdot]$ we only need to know $(c_{ij}^k)_{ijk=1}^n$, which are called the structural constants of L.

Exercise. What are the restrictions on (c_{ij}^k)?

II Homomorphisms, Subalgebras, Ideals

Definition II.1. Let L_1, L_2 be Lie algebras over F, and let $\phi\colon L_1 \to L_2$ be a map such that

1. ϕ is linear
2. $\forall x, y \in L_1$, $[\phi(x), \phi(y)]_{L_2} = \phi([x, y]_{L_1})$

then ϕ is called a *Lie algebras homomorphism.* If additionally ϕ is a bijection then we say ϕ is a *Lie algebras isomorphism*, and that L_1 and L_2 are isomorphic (written $L_1 \cong L_2$).

Lemma II.2. *Let L_1, L_2 be Lie algebras over the same field F. Then the following are equivalent:*

- $L_1 \cong L_2$
- $\exists$ *bases B_1, B_2 of L_1, L_2 resp. such that the structural constants of L_1 w.r.t. B_1 are equal to the structural constants of L_2 w.r.t. B_2.*

Proof. Exercise. ■

Examples.

1. $\phi\colon L \to L$, $\phi(x) = x$.
2. $\phi\colon L \to L$, $\phi(x) = 0$.
3. $L_1 = \mathfrak{gl}(n, F)$,
   $L_2 = (F, 0_{F\times F})$ (F as an abelian Lie algebra over itself),
   $\operatorname{tr}\colon \mathfrak{gl}(n, F) \to F$ is linear and $\operatorname{tr}([A, B]_{L_1}) = \operatorname{tr}(AB - BA) = 0 = [\operatorname{tr} A, \operatorname{tr} B]_{L_2}$ so tr is a homomorphism.
4. $L_1 = \mathfrak{gl}(V)$, where $\dim_F V = n$,
   $L_2 = \mathfrak{gl}(n, F)$,
   fix a basis $(e_1, \ldots, e_n)$ and now for all linear $T\colon V \to V$ there is a unique matrix A representing T w.r.t. $(e_1, \ldots, e_n)$. Define $\phi\colon \mathfrak{gl}(V) \to \mathfrak{gl}(n, F)$ by $\phi(T) = A$.
   ϕ is a vector space isomorphism, and $\phi([T_1, T_2]) = \phi(T_1 \circ T_2 - T_2 \circ T_1) = A_1A_2 - A_2A_1 = [A_1, A_2] = [\phi(T_1), \phi(T_2)]$. Hence ϕ is a Lie algebras isomorphism, and $\mathfrak{gl}(V) \cong \mathfrak{gl}(n, F)$.

Definition II.3. Let $K \subseteq L$. If

1. K is a subspace of L, and
2. $\forall x, y \in K$, $[x, y] \in K$,

then K is a *subalgebra* of L.

Examples.

1. $\mathfrak{sl}(n, F)$ is a subalgebra of $\mathfrak{gl}(n, F)$.
2. $\mathfrak{b}(n, F)$ is a subalgebra of $\mathfrak{gl}(n, F)$.

3. $\mathfrak{u}(n,F)$ is a subalgebra of $\mathfrak{gl}(n,F)$.
   $\mathfrak{u}(n,F)$ is a subalgebra of $\mathfrak{sl}(n,F)$.
   $\mathfrak{u}(n,F)$ is a subalgebra of $\mathfrak{b}(n,F)$.

4. $\mathfrak{sp}(n,F)$ is a subalgebra of $\mathfrak{gl}(n,F)$.
   $\mathfrak{sp}(n,F)$ is a subalgebra of $\mathfrak{sl}(n,F)$.

Definition II.4. Let $I \subseteq L$. If

1. I is a subspace of L, and

2. $\forall x \in I,\ y \in L,\ [x,y] \in I$,

then I is an *ideal* of L.

Since $[x,y] = -[y,x]$, there is no difference between left and right ideals for Lie algebras.

General Examples of Ideals, Subalgebras, Homomorphisms

1. (a) L is a subalgebra and an ideal of L.
   (b) $\{0\}$ is a subalgebra and an ideal of L.
   (c) Let $x \in L$, then $\langle x \rangle = Fx$ is an abelian subalgebra of L.
   (d) $Z(L) = \{z \in L \mid \forall x \in L,\ [x,z] = 0\}$ is the center of L.
   $Z(L)$ is abelian and an ideal of L.

   Exercise. Find $Z(\mathfrak{gl}(2,F))$, $Z(\mathfrak{sl}(2,F))$, $Z(\mathfrak{sp}(2,F))$ and $Z(\mathfrak{o}(2,F))$.

2. Let $\phi\colon L_1 \to L_2$ be a Lie algebras homomorphism.

   Lemma II.5. $\operatorname{im}\phi$ *is a subalgebra of* L_2*, and* $\ker\phi$ *is an ideal of* L_1*.*

   Proof. Exercise. ∎

   Example. $\operatorname{tr}\colon \mathfrak{gl}(n,F) \to F$ is surjective, $\ker\operatorname{tr} = \mathfrak{sl}(n,F)$, hence $\mathfrak{sl}(n,F)$ is an ideal of $\mathfrak{gl}(n,F)$.

3. Let L be a Lie algebra over F, then L is a vector space over F, hence we can consider $\mathfrak{gl}(L)$. Define $\operatorname{ad}\colon L \to \mathfrak{gl}(L)$ by $\operatorname{ad}(x)(y) = [x,y]$. Then ad is a Lie algebras homomorphism, and $\ker(\operatorname{ad}) = Z(L)$.

More About Ideals

Definition II.7. Let L be a Lie algebra over F and let I, J be ideals of L. Then:

$$
\begin{aligned}
I + J :=& \{i + j \mid i \in I,\ j \in J\} \\
[I,J] :=& \langle [i,j] \mid i \in I,\ j \in J \rangle \\
=& \left\{ \sum_{k=1}^{l} c_k [i_k, j_k] \mid c_k \in F,\ i_k \in I,\ j_k \in J,\ l \in \mathbb{N} \right\}
\end{aligned}
$$

Lemma II.8. *If L is a Lie algebra, and I, J are ideals, then the following are also ideals:*

1. $I \cap J$
2. $I + J$
3. $[I, J]$

Proof.

1. easy.
2. $I + J$ is a subspace of L. For $x \in L$, $y \in I + J$ we have $y = i + j$ for some $i \in I$, $j \in J$ and so $[x, y] = [x, i + j] = [x, i] + [x, j] \in I + J$.
3. $[I, J]$ is a subspace of L by its definition as a span. Now $\forall x \in L$, $y \in [I, J]$, $\exists c_1, \dots, c_l \in F$, $i_1, \dots, i_l \in I$, $j_1, \dots, j_l \in J$ s.t. $y = \sum_{k=1}^{l} c_k [i_k, j_k]$.
   Now $[x, y] = \left[x, \sum_{k=1}^{l} c_k [i_k, j_k]\right] = \sum_{k=1}^{l} c_k [x, [i_k, j_k]]$. But $\forall k$, $[x, [i_k, j_k]] = -[j_k, [x, i_k]] - [i_k, [j_k, x]] \in [I, J]$ since $[x, i_k] \in I$ and $[j_k, x] \in J$. ■

Example. Let L be a Lie algebra, then $[L, L]$ is an ideal of L.

Exercise. If $L = \mathfrak{gl}(2, F)$, what is $[L, L]$?
If $L = \mathfrak{sl}(2, F)$, what is $[L, L]$?

Quotient Lie Algebras

Let V be a vector space over F, let W be a subspace of V. A coset of W in V with representative $v \in V$ is

$$v + W = \{v + w \mid w \in W\}.$$

For $v, \widetilde{v} \in V$, $v + W = \widetilde{v} + W \iff v - \widetilde{v} \in W$, and either $v + W = \widetilde{v} + W$ or $(v + W) \cap (\widetilde{v} + W) = \emptyset$. We can now define

$$V/W := \{v + W \mid v \in V\}$$

$\forall x + W, y + W \in V/W$ define $(x + W) \oplus (y + W) := (x + y) + W$.
$\forall \alpha \in F$, $\forall x + W \in V/W$ define $\alpha \cdot (x + W) := (\alpha x) + W$.
This makes $(V/W, \oplus, \cdot)$ a vector space over F.

Now let L be a Lie algebra over F, and I an ideal. We define as above the quotient (vector) space L/I. Now we define $[\cdot, \cdot] \colon L/I \times L/I \to L/I$ by $[x + I, y + I] = [x, y]_L + I$. To check that this is well defined, let $v_1 + I = \widetilde{v_1} + I$, $v_2 + I = \widetilde{v_2} + I$, i.e. $v_1 - \widetilde{v_1} = i_1 \in I$, $v_2 - \widetilde{v_2} = i_2 \in I$. Now

$$\begin{aligned}
[v_1 + I, v_2 + I] &= [v_1, v_2] + I = [\widetilde{v_1} + i_1, \widetilde{v_2} + i_2] + I \\
&= [\widetilde{v_1}, \widetilde{v_2}] + \underbrace{[\widetilde{v_1}, i_2] + [i_1, \widetilde{v_2}] + [i_1, i_2]}_{\in I} + I \\
&= [\widetilde{v_1}, \widetilde{v_2}] + I \\
&= [\widetilde{v_1} + I, \widetilde{v_2} + I] + I
\end{aligned}$$

and so $[\cdot,\cdot]$ is well-defined. Exercise: show that it is a Lie bracket. Thus $(L/I, \oplus, \cdot, [\cdot,\cdot])$ is a Lie algebra over F.

Let L be a Lie algebra over F and I an ideal. Define $\pi\colon L \to L/I$ by $\pi(x) = x + I$.
$\forall x, y \in L$, $\pi(x+y) = (x+y)+I = (x+I) \oplus (y+I) = \pi(x) \oplus \pi(y)$.
$\forall \alpha \in F$, $\forall x \in L$, $\pi(\alpha x) = (\alpha x) + I = \alpha \cdot (x+I) = \alpha \cdot \pi(x)$.
$\therefore$ π is linear.
$\forall x, y \in L$, $\pi([x,y]) = [x,y] + I = [x+I, y+I] = [\pi(x), \pi(y)]$.
$\therefore$ π is a Lie algebras homomorphism.

1st Isomorphism Theorem. *Let $\phi\colon L_1 \to L_2$ be a Lie algebras homomorphism, then:*

1. $\operatorname{im}\phi$ *is a subalgebra of* L_2
2. $\ker\phi$ *is an ideal of* L_1
3. $L_1/\ker\phi \cong \operatorname{im}\phi$

Proof.

1. 2. Lemma II.5.

3. Let $I = \ker\phi$, define $f\colon L_1/I \to \phi(L_1)$ by $f(x+I) = \phi(x)$.
To show this is well-defined, let $x + I = \widetilde{x} + I$, i.e. $x - \widetilde{x} = i \in I$. Now

$$\begin{aligned} f(x+I) - f(\widetilde{x}+I) &= \phi(x) - \phi(\widetilde{x}) \\ &= \phi(x - \widetilde{x}) = \phi(i) \\ &= 0 \end{aligned}$$

$\therefore$ f is well-defined.
$\forall x+I, y+I \in L_1/I$,

$$\begin{aligned} f((x+I)\oplus(y+I)) &= f((x+y)+I) = \phi(x+y) = \phi(x)+\phi(y) \\ &= f(x+I) + f(y+I) \end{aligned}$$

$\forall \alpha \in F$, $\forall x + I \in L_1/I$,

$$\begin{aligned} f(\alpha\cdot(x+I)) &= f(\alpha x + I) = \phi(\alpha x) = \alpha\phi(x) \\ &= \alpha f(x+I) \end{aligned}$$

$\therefore$ f is linear.
$\forall x+I, y+I \in L_1/I$,

$$\begin{aligned} f([x+I, y+I]) &= f([x,y]+I) = \phi([x,y]) = [\phi(x), \phi(y)] \\ &= [f(x+I), f(y+I)] \end{aligned}$$

$\therefore$ f preserves the Lie bracket and so is a Lie algebras homomorphism.
$\forall z \in \phi(L_1)$, $\exists v \in L_1$ s.t. $z = \phi(v) = f(v+I)$.
$\forall v + I \in L_1/I$, $f(v+I) \Rightarrow f(v+I) = \phi(v) = 0 \Rightarrow v + I = 0 + I$.
$\therefore$ f is bijective, and hence a Lie algebras isomorphism. Hence $L_1/\ker\phi \cong \operatorname{im}\phi$. ∎

2nd Isomorphism Theorem. *Let I and J be ideals of a Lie algebra L, then:*

1. $I + J$ *is an ideal of* L
2. $I \cap J$ *is an ideal of* L
3. $(I + J)/J \cong I/(I \cap J)$

3rd Isomorphism Theorem. *Let I and J be ideals of a Lie algebra L with $I \subseteq J$, then:*

1. J/I *is an ideal of* L/I
2. $(L/I)/(J/I) \cong L/J$

Correspondence Theorem. *Let I be an ideal of a Lie algebra L, then there is a bijection between the following sets:*

- $\{J \mid J \text{ is an ideal of } L \text{ with } J \supseteq L\}$
- $\{K \mid K \text{ is an ideal of } L/I\}$

Some Constructions

Let L_1, L_2 be Lie algebras over F. Then define:

$$\begin{aligned}
&L_1 \oplus L_2 := \{(a_1, a_2) \mid a_1 \in L_1,\ a_2 \in L_2\} = L_1 \times L_2 \\
&\forall (a_1, a_2), (b_1, b_2) \in L_1 \oplus L_2,\ (a_1, a_2) + (a_1, a_2) := (a_1 + b_1, a_2 + b_2) \\
&\forall \alpha \in F,\ \forall (a_1, b_1) \in L_1 \oplus L_2,\ \alpha(a_1, a_2) := (\alpha a_1, \alpha a_2) \\
&\forall (a_1, a_2), (b_1, b_2) \in L_1 \oplus L_2,\ [(a_1, a_2), (a_1, a_2)] := ([a_1, b_1], [a_2, b_2])
\end{aligned}$$

With these operations, $L_1 \oplus L_2$ is a Lie algebra over F.

Lemma II.9. *Let L_1, L_2 be Lie algebras over F, then:*

1. $Z(L_1 \oplus L_2) = Z(L_1) \oplus Z(L_2)$
2. $[L_1 \oplus L_2, L_1 \oplus L_2] = [L_1, L_1] \oplus [L_2, L_2]$

Proof. Exercise. ■

Exercise. $\mathfrak{gl}(2, \mathbb{C}) \cong \mathfrak{sl}(2, \mathbb{C}) \oplus \mathbb{C}$.

$$\therefore Z(\mathfrak{gl}(2, \mathbb{C})) \cong Z(\mathfrak{sl}(2, \mathbb{C})) \oplus \overbrace{Z(\mathbb{C})}^{\mathbb{C}}$$
$$\therefore [\mathfrak{gl}(2, \mathbb{C}), \mathfrak{gl}(2, \mathbb{C})] \cong [\mathfrak{sl}(2, \mathbb{C}), \mathfrak{sl}(2, \mathbb{C})] \oplus \underbrace{[\mathbb{C}, \mathbb{C}]}_{0}$$

An F-algebra is a vector space over F with a bilinear operation $(a, b) \mapsto a \cdot b$. Let U be an F-algebra, then a map $\delta \colon U \to U$ is a derivation of U if:

1. δ is linear
2. $\forall a, b \in U$, $\delta(a \cdot b) = a \cdot \delta(b) + \delta(a) \cdot b$.

Now consider $\mathrm{Der}(U) := \{\delta \mid \delta \text{ is a derivation of } U\}$.
$\forall \delta_1, \delta_2 \in \mathrm{Der}(U)$, $\delta_1 + \delta_2 \colon U \to U$ is a derivation since it is linear and

$$\begin{aligned}(\delta_1 + \delta_2)(a \cdot b) &= \delta_1(a \cdot b) + \delta_2(a \cdot b) = a \cdot \delta_1(b) + \delta_1(a) \cdot b + a \cdot \delta_2(b) + \delta_2(a) \cdot b \\ &= a \cdot ((\delta_1 + \delta_2)(b)) + ((\delta_1 + \delta_2)(a)) \cdot b\end{aligned}$$

$\forall \alpha \in F$, $\forall \delta \in \mathrm{Der}(U)$, $\alpha\delta \colon U \to U$ is a derivation since it is linear and

$$\begin{aligned}(\alpha\delta)(a \cdot b) &= \alpha(\delta(a \cdot b)) = \alpha(a \cdot \delta(b) + \delta(a) \cdot b) \\ &= a \cdot \alpha\delta(b) + \alpha\delta(a) \cdot b\end{aligned}$$

So $\mathrm{Der}(U)$ with these operations is a vector space over F.

$\forall \delta_1, \delta_2 \in \mathrm{Der}(U)$ define $[\delta_1, \delta_2] := \delta_1 \circ \delta_2 - \delta_2 \circ \delta_1$
$[\delta_1, \delta_2]$ is linear, is it a derivation? Let $a, b \in U$, then:

$$\begin{aligned}[\delta_1, \delta_2](a \cdot b) &= \delta_1(\delta_2(a \cdot b)) - \delta_2(\delta_1(a \cdot b)) \\ &= \delta_1(a \cdot \delta_2(b) + \delta_2(a) \cdot b) - \delta_2(a \cdot \delta_1(b) + \delta_1(a) \cdot b) \\ &= [a \cdot \delta_1(\delta_2(b)) + \delta_1(a) \cdot \delta_2(b) + \delta_2(a) \cdot \delta_1(b) + \delta_1(\delta_2(a)) \cdot b] \\ &\quad - [a \cdot \delta_2(\delta_1(b)) + \delta_2(a) \cdot \delta_1(b) + \delta_1(a) \cdot \delta_2(b) + \delta_2(\delta_1(a)) \cdot b] \\ &= a \cdot [\delta_1(\delta_2(b)) - \delta_2(\delta_1(b))] + [\delta_1(\delta_2(a)) - \delta_2(\delta_1(a))] \cdot b \\ &= a \cdot [\delta_1, \delta_2](b) + [\delta_1, \delta_2](a) \cdot b\end{aligned}$$

Hence $[\delta_1, \delta_2] \in \mathrm{Der}(U)$. Additionally $[\cdot, \cdot]$ is a Lie bracket since it is a restriction of the Lie bracket of $\mathfrak{gl}(U)$, and so $(\mathrm{Der}(U), [\cdot, \cdot])$ is a Lie algebra over F, and also a subalgebra of $\mathfrak{gl}(U)$.

If L is a Lie algebra over F, then L is, in fact, an F-algebra! Thus we can consider $\mathrm{Der}(L) = \{\delta \colon L \to L \mid \delta \text{ is linear, } \delta([a, b]) = [a, \delta(b)] + [\delta(a), b]\}$ which is a Lie subalgebra of $\mathfrak{gl}(L)$. Given $x \in L$, we define $\mathrm{ad}(x) \colon L \to L$ by $\mathrm{ad}(x)(y) = [x, y]$. $\mathrm{ad}(x)$ is linear and $\forall a, b \in L$:

$$\begin{aligned}\mathrm{ad}(x)([a, b]) &= [x, [a, b]] \\ &= -[b, [x, a]] - [a, [b, x]] \\ &= [[x, a], b] + [a, [x, b]] \\ &= [\mathrm{ad}(x)(a), b] + [\mathrm{ad}(x)(b)]\end{aligned}$$

and so $\mathrm{ad}(x) \in \mathrm{Der}(L)$. Now $\mathrm{ad} \colon L \mapsto \mathrm{ad}(L) \subseteq \mathrm{Der}(L) \subseteq \mathfrak{gl}(L)$ is called the *adjoint homomorphism*, elements of $\mathrm{ad}(L)$ are called *inner derivations* and elements of $\mathrm{Der}(L) \setminus \mathrm{ad}(L)$ are called *outer derivations*.

III Solvable Lie Algebras

Lemma III.1. *Let L be a Lie algebra, and I be an ideal of L. Then L/I is abelian iff $I \supseteq [L, L]$.*

Proof.

$$\begin{aligned} L/I \text{ is abelian} &\iff \forall x+I, y+I \in L/I,\ [x+I, y+I] = I \\ &\iff \forall x, y \in L,\ [x, y] \in I \\ &\iff [L, L] \subseteq I \end{aligned}$$

■

Corollary III.2. *$L/[L, L]$ is abelian, and $[L, L]$ is the smallest ideal giving an abelian quotient.*

Definition III.3. Let L be a Lie algebra over F. A *derived series* of L is $L^{(0)}, L^{(1)}, L^{(2)}, \ldots$ where:

$$L^{(0)} := L$$

$$L^{(k+1)} := [L^{(k)}, L^{(k)}]$$

Remark: $L^{(0)} \supseteq L^{(1)} \supseteq L^{(2)} \supseteq \cdots$ and each $L^{(i)}$ is an ideal of L, since the bracket of two ideals is an ideal.

Definition III.4. L is *solvable* if $\exists n \in \mathbb{N}$ s.t. $L^{(n)} = \{0\}$

Examples.

1. If L is abelian then $L^{(1)} = [L, L] = \{0\}$ and therefore L is solvable.
2. $L = b(2, \mathbb{C}) = \langle e_{11}, e_{12}, e_{22} \rangle_{\mathbb{C}}$
   hence $L^{(1)} = [L, L] = \langle [e_{11}, e_{12}], [e_{11}, e_{22}], [e_{12}, e_{22}] \rangle_{\mathbb{C}} = \langle e_{12} \rangle_{\mathbb{C}} = u(2, \mathbb{C})$
   hence $L^{(2)} = \{0\}$ and so $b(2, \mathbb{C})$ is solvable. Exercise: $b(n, \mathbb{C})$ is solvable.
3. $L = \mathfrak{sl}(2\mathbb{C})$: $L^{(1)} = [L, L] = L$, $L^{(0)} = L^{(1)} = L^{(2)} = \cdots$ and so $\mathfrak{sl}(2, \mathbb{C})$ is not solvable. Exercise: $\mathfrak{sl}(n, \mathbb{C})$ is not solvable.

Lemma III.6. *Let $\phi\colon L_1 \to L_2$ be a surjective Lie algebras homomorphism. Then $\phi(L_1^{(i)}) = L_2^{(i)}$.*

Proof. $\phi(L_1^{(0)}) = L_2^{(0)}$
Now suppose $\phi(L_1^{(k)}) = L_2^{(k)}$, then:

$$\begin{aligned} \phi(L_1^{(k+1)}) &= \phi([L_1^{(k)}, L_1^{(k)}]) = [\phi(L_1^{(k)}), \phi(L_1^{(k)})] = [L_2^{(k)}, L_2^{(k)}] \\ &= L_2^{(k+1)} \end{aligned}$$

The result follows by induction. ■

Proposition III.7. *Let L be a Lie algebra, then:*

1. *If L is solvable, then so is every subalgebra of L.*
2. *If L is solvable, then so is every homomorphic image of L.*
3. *If I is an ideal of L and both I and L/I are solvable, then so is L.*

4. *If I and J are solvable ideals, then so is $I + J$.*

Proof.

1. L is solvable, M is a subalgebra of L. $\forall i \in \mathbb{N}$, $M^{(i)} \subseteq L^{(i)}$. Hence $L^{(m)} = \{0\} \Rightarrow M^{(m)} = \{0\}$.

2. Let $\phi\colon L \to \widetilde{L}$ be a Lie algebras homomorphism. By III.6, $\forall i \in \mathbb{N}$, $\phi(L^{(i)}) = (\phi(L))^{(i)}$. Hence $L^{(m)} = \{0\} \Rightarrow (\phi(L))^{(m)} = \phi(L^{(m)}) = \phi(\{0\}) = \{0\}$.

3. L is a Lie algebra, I is an ideal, I and L/I are solvable.
   Consider $\pi\colon L \to L/I$

   $$(L^{(i)} + I)/I = \pi(L^{(i)}) = (\pi(L))^{(i)} = (L/I)^{(i)}$$

   Since L/I is solvable, $\exists m \in \mathbb{N}$ s.t.:

   $$\begin{aligned} (L/I)^{(m)} &= \{0_{L/I}\} = I/I \\ \therefore I/I &= (L^{(m)} + I)/I \\ \therefore I &= L^{(m)} + I \\ \therefore L^{(m)} &\subseteq I \end{aligned}$$

   Since I is solvable, $\exists n \in \mathbb{N}$ s.t.:

   $$\begin{aligned} I^{(n)} &= \{0\} \\ \therefore (L^{(m)})^{(n)} &\subseteq I^{(n)} = \{0\} \\ \therefore L^{(m+n)} &= (L^{(m)})^{(n)} = \{0\} \end{aligned}$$

   Therefore L is solvable.

4. I, J are solvable ideals of L, so $I+J$ is an ideal of L and $(I+J)/I \cong J/(I \cap J)$ by the 2nd Isomorphism theorem. J is solvable so by 2., $J/(I \cap J) = \pi(J)$ is solvable. Thus $(I + J)/I$ is solvable, and I is solvable. Therefore by 3., $I + J$ is solvable. ■

Let L be a Lie algebra. Consider $\mathfrak{a} = \{I \mid I \text{ is a solvable ideal of } L\}$ and choose $R \in \mathfrak{a}$ of maximal dimension. Now $\forall I \in \mathfrak{a}$, $R + I$ is a solvable ideal, hence $R + I \in \mathfrak{a}$. But now $\dim R + I \leqslant \dim R \leqslant \dim R + I$ since $R \subseteq R + I$ and R has maximal dimension. Thus $R + I = R$ and hence $I \subseteq R$. So there exists a unique solvable ideal containing every other solvable ideal of L. We call this the radical ideal of L and denote it $\mathrm{Rad}(L)$.

Definition III.8. A Lie algebra L is called *semisimple* if $\mathrm{Rad}(L) = \{0\}$.

Examples.

1. $L = b(2, \mathbb{C})$ is not semisimple since $\mathrm{Rad}(L) = L$.

2. $L = \mathfrak{sl}(2, \mathbb{C})$ is semisimple (excercise).

3. $L = \mathfrak{gl}(2, \mathbb{C}) \cong \mathfrak{sl}(2, \mathbb{C}) \oplus \mathbb{C}$, hence $\mathrm{Rad}(L) = \{0\} \oplus \mathbb{C}$, hence L is not semisimple

Lemma III.9. *If L is a Lie algebra, then $L/\operatorname{Rad}(L)$ is semisimple.*

Proof. Let $\bar{L} = L/\operatorname{Rad}(L)$. Let K be a solvable ideal of $\bar{L}$. By the Correspondence Theorem there is an ideal J of L s.t. $J \supseteq \operatorname{Rad}(L)$ and $J/\operatorname{Rad}(L) = K$. Since $J/\operatorname{Rad}(L) = K$ is solvable and $\operatorname{Rad}(L)$ is solvable:
J is solvable, hence $J = \operatorname{Rad}(L)$.
$\therefore K = \operatorname{Rad}(L)/\operatorname{Rad}(L) = \{0\}$
$\therefore \operatorname{Rad}(\bar{L}) = \{0\}$
$\therefore L/\operatorname{Rad}(L)$ is semisimple. ■

IV Nilpotent Lie Algebras

Definition IV.1. Let L be a Lie algebra. A *central series* of L is $L^0, L^1, L^2, \dots$ where:

$$L^0 := L$$

$$L^{k+1} := [L, L^k]$$

Remark: $L^0 \supseteq L^1 \supseteq L^2 \supseteq \cdots$, and each L^i is an ideal of L.

Definition IV.2. L is *nilpotent* if $\exists m \in \mathbb{N}$ s.t. $L^m = \{0\}$.

$$[L/L^{k+1}, L^k/L^{k+1}] = ([L, L^k] + L^{k+1})/L^{k+1} = (L^{k+1} + L^{k+1})/L^{k+1} = \{0\}$$

So $L^k/L^{k+1} \subseteq Z(L/L^{k+1})$.

Examples.

1. $L = \mathfrak{sl}(2, \mathbb{C})$, $[L, L] = L$ so $\forall n \in \mathbb{N}$, $L^n = L$, hence L is not nilpotent.

2. $L = b(2, \mathbb{C}) = \langle e_{11}, e_{12}, e_{22} \rangle$
   $L^1 = u(2, \mathbb{C}) = \langle e_{12} \rangle$
   $L^2 = [L, L^1] = \langle [e_{11}, e_{12}], [e_{12}, e_{12}], [e_{22}, e_{12}] \rangle$
   $L^2 = \langle e_{12} \rangle = L^1$
   Similarly $L^k = L^1$ for all $k \geqslant 1$ and so L is not nilpotent.

3. $L = u(2, \mathbb{C}) = e_{12}$, $L^1 = \{0\}$ so L is nilpotent.

3'. $L = u(3, \mathbb{C})$ is the Heisenberg Lie algebra, $L = \langle e_{12}, e_{13}, e_{23} \rangle_{\mathbb{C}}$.
   $L^1 = \langle \underbrace{[e_{12}, e_{13}]}_{=0}, \underbrace{[e_{12}, e_{23}]}_{=e_{13}}, \underbrace{[e_{13}, e_{23}]}_{=0} \rangle_{\mathbb{C}} = \langle e_{13} \rangle_{\mathbb{C}}$.
   $L^2 = [L, L^1] = \langle [e_{12}, e_{13}], [e_{13}, e_{13}], [e_{23}, e_{13}] \rangle = \{0\}$.
   So L is nilpotent.

4. If L is abelian then L is nilpotent since $L^1 = [L, L] = \{0\}$.

Lemma IV.3. *If L is nilpotent then L is solvable.*

Proof. Show by induction that $L^{(n)} \subseteq L^n$: $L^0 = L = L^{(0)}$.
Assume $L^{(k)} \subseteq L^k$, then $L^{(k+1)} = [L^{(k)}, L^{(k)}] \subseteq [L, L^{(k)}] \subseteq [L, L^k] = L^{k+1}$.
Thus L is nilpotent $\Rightarrow \exists n \in \mathbb{N}$ s.t. $L^n = \{0\} \Rightarrow \exists n \in \mathbb{N}$ s.t. $L^{(n)} = \{0\} \Rightarrow L$ is solvable. ■

Proposition IV.4. *Let L be a Lie algebra over F. Then:*

1. *If L is nilpotent, then so is every subalgebra of F.*

2. *If $L \neq \{0\}$ is nilpotent, then $Z(L) \neq \{0\}$.*

3. *If $L/Z(L)$ is nilpotent, then so is L.*

Proof.

1. If M is a subalgebra of L, then $M^i \subseteq L^i$, hence $L^n = \{0\} \Rightarrow M^n = \{0\}$.

2. $Z(L) = \{x \in L \mid \forall y \in L,\ [x,y] = 0\}$. If L is nilpotent then $\exists n \in \mathbb{N}$ s.t. $L^n = \{0\} \neq L^{n-1}$. But $L^n \subseteq Z(L)$ since $[L, L^n] = L^{n+1} = \{0\}$.

3. If $\bar{L} = L/Z(L)$ is nilpotent, then $\exists n \in \mathbb{N}$ s.t. $\bar{L}^n = \{0_{\bar{L}}\} = Z(L)/Z(L)$. But $\bar{L}^n = [\bar{L}, \bar{L}^{n-1}] = [L/Z(L), (L^{n-1} + Z(L))/Z(L)] = ([L, L^{n-1}] + Z(L))/Z(L)$. Thus $[L, L^{n-1}] \subseteq Z(L)$ and so $L^{n+1} \subseteq [Z, Z(L)] = \{0\}$. ■

Let L be a nilpotent Lie algebra, then there is an integer $n \in \mathbb{N}$ such that $L^n = \{0\}$.
We then have $\{0\} = [L, L^{n-1}] = [L, [L, L^{n-2}]] = [L, [L, \ldots [L, L] \ldots]]$, and hence for all x and y in L, $[x, [x, \ldots [x, y] \ldots]] = 0$, so that $\mathrm{ad}(x)\,\mathrm{ad}(x) \cdots \mathrm{ad}(x)(y) = 0$ and $\mathrm{ad}(x)^n(y) = 0$ for all $x, y \in L$. Therefore $\mathrm{ad}(x)^n$, considered as a linear map $L \to L$, is just the zero map.

Definition IV.5. If L is a Lie algebra and $x \in L$, then x is ad-*nilpotent* if and only if there exist an integer $i_x \in \mathbb{N}$ such that $\mathrm{ad}(x)^{i_x} = 0$ as a linear map $L \to L$.

Theorem. *If L is a nilpotent Lie algebra, then all $x \in L$ are* ad-*nilpotent.*

The converse will take some time to prove, but is stated here as a goal:

Engel's Theorem. *Let L be a Lie algebra such that x is* ad-*nilpotent for all $x \in L$. Then L is nilpotent.*

Definition IV.6. $x\colon V \to V$ is a *nilpotent* linear map if and only if there is an integer $i \in \mathbb{N}$ such that x^i is the 0 map $0\colon V \to V$.

Lemma IV.7. *If $x \in L \subseteq \mathfrak{gl}(V)$ is nilpotent ($x\colon V \to V$), then x is* ad-*nilpotent ($\mathrm{ad}(x)\colon L \to L$).*

Proof. For all $y \in L$, we have that $\mathrm{ad}(x)(y) = [x, y] = x \circ y - y \circ x$, and then $\mathrm{ad}(x)^2(y) = [x, [x, y]] = [x, x \circ y - y \circ x] = x^2 \circ y - 2x \circ y \circ x + y \circ x^2$. In general, we have the following formula:

$$\mathrm{ad}(x)^k(y) = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} x^i \circ y \circ x^{k-i}$$

If x is nilpotent, then $x^j = 0$ for some j, so that if we choose $k > 2j$, the above formula is always zero. This proves the lemma. ■

Let V be a vector space over F of dimension n, and $A\colon V \to V$ be a linear map. $v \in V \setminus \{0\}$ is an eigenvector of A if and only if there exists $\lambda \in F$ such that $A(v) = \lambda v$.
Now, consider $L \subseteq \mathfrak{gl}(V)$ and a subalgebra M of L. v is a common eigenvector for all elements of M if and only if $\forall A \in M,\ \exists \lambda_A \in F$ such that $A(v) = \lambda_A v$. This gives us a function $\lambda\colon M \to F$, given by $\lambda\colon A \mapsto \lambda_A$.
Then $v \in V \setminus \{0\}$ is an eigenvector for M is equivalent to the existence of a function $\lambda\colon M \to F$ such that $A(v) = \lambda(A)v$ for all $A \in M$. Let us look at all eigenvectors for M with given function λ.
Set $V_\lambda = \{w \in V : A(w) = \lambda(A)w\ \forall A \in M\} \ni v$

Question. What can we say about V_λ?

1. $V_\lambda \neq \{0\}$ as $v \in V_\lambda$.
2. For all $w_1, w_2 \in V_\lambda$, for all $\alpha_1, \alpha_2 \in F$ and for all $A \in M$, we have that

$$A(\alpha_1 w_1 + \alpha_2 w_2) = \alpha_1 A(w_1) + \alpha_2(w_2) = \lambda(A)\alpha_1 w_1 + \lambda(A)\alpha_2 w_2$$

so that V_λ is a (nontrivial) subspace of V.

On the other hand, we can look at $(\gamma_1 A_1 + \gamma_2 A_2)(v) = \gamma_1 A_1(v) + \gamma_2 A_2(v)$, which is clearly equal to $\gamma_1\lambda(A_1)v + \gamma_2\lambda(A_2)v$. But $\gamma_1 A_1 + \gamma_2 A_2$ is an element of M, so that $(\gamma_1 A_1 + \gamma_2 A_2)(w) = \lambda(\gamma_1 A_1 + \gamma_2 A_2)w$ and hence $\gamma_1\lambda(A_1) + \gamma_2\lambda(A_2) = \lambda(\gamma_1 A_1 + \gamma_2 A_2)$, therefore λ is linear.

Definition IV.8. Let M be a subalgebra of $\mathfrak{gl}(V)$. A *weight* of M is a linear map $\lambda\colon M \to F$ such that $\{0\} \neq V_\lambda = \{A(w) = \lambda(A)w \ \forall A \in M\}$. V_λ is a subspace of V.
In this case V_λ is a *weight subspace* associated to the weight λ.

Remark. If λ is a weight of M then $V_\lambda \neq \{0\}$ implies that there exists an eigenvector for M.

Example. $L = \mathfrak{b}(2, \mathbb{C}) = \langle e_{11}, e_{22}, e_{12}\rangle_{\mathbb{C}} \subseteq \mathfrak{gl}(2, \mathbb{C})$.
Then take $v = \begin{pmatrix}1\\0\end{pmatrix}$, we get the following:

$$e_{11}v = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} = 1v$$

$$e_{22}v = \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = 0v$$

$$e_{12}v = \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = 0v$$

So define $\lambda\colon \mathfrak{b}(2, \mathbb{C}) \to \mathbb{C}$ by $\lambda(e_{11}) = 1$, $\lambda(e_{12}) = \lambda(e_{22}) = 0$ and extend by linearity.

Exercise. Calculate V_λ.

Lemma IV.9. *Let $I \subseteq L = \mathfrak{gl}(V)$ be an ideal and $W = \{w \in V : A(w) = 0 \ \forall A \in I\}$. If $W \neq \{0\}$ then $W = V_0$ is the weight subspace associated to $\lambda = 0$ and V_0 is L-invariant (a space $W \subseteq V$ is L-invariant if for all $x \in L$, $w \in W \Rightarrow x(w) \in W$).*

Lemma IV.10. *Let L be a Lie subalgebra of $\mathfrak{gl}(V)$ where V is a vector space over F of characteristic* 0. *Let I be an ideal of L and λ a weight of I. Then V_λ is L-invariant.*

Proof. I is an ideal of $L \subseteq \mathfrak{gl}(V)$, $\lambda\colon I \to F$ is linear and $V_\lambda = \{w \in V : A(w) = \lambda(A)w \ \forall A \in I\}$.
We want to show V_λ is L-invariant, but this is true if and only if for all $x \in L$ and for all $w \in V_\lambda$ we have that $x(w) \in V_\lambda$, which is equivalent to showing that $A(x(w)) = \lambda(A)x(w)$ for all $x \in L$, all $w \in V_\lambda$ and all $A \in I$.

Note that, taking $x \in L \subseteq \mathfrak{gl}(V)$ and $A \in I$, we have that $[x, A] = xA - Ax$. Now, for all $w \in V_\lambda$ we have:

$$\begin{aligned}\lambda([x, A])w &= [x, A](w) \\ &= xA(w) - Ax(w) \\ &= x(\lambda(a)w) - A(x, w)\end{aligned}$$

so that $A(x(w)) = \lambda(A)x(w) - \lambda([x, A])w$. It remains to show that $\lambda([x, A]) = 0$ for all $x \in L$ and all $A \in I$.
For all $x \in L$ and all $w \in V_\lambda$, consider $W = \langle w, x(w), x^2(w), \ldots \rangle$. Since $\dim_F(V) < \infty$, we have that $\dim_F(W) < \infty$, say $\dim_F(W) = m$, and then $w, x(w), \ldots, x^{m-1}(w)$ is a basis of W.
Now for all $y \in I$, $y(w) = \lambda(y)w \in W$, and we then have

$$\begin{aligned}y(x(w)) &= y(x(w)) \\ &= ([y, x] + xy)(w) \\ &= [y, x](w) + x(y(w)) \\ &= \lambda([y, x])w + x(\lambda(y)w) \\ &= \lambda([y, x])w + \lambda(y)x(w)\end{aligned}$$

which is in W. Also

$$\begin{aligned}y(x^2(w)) &= yx(x(w)) \\ &= (xy + [y, x])(x(w)) \\ &= xy(x(w)) + [y, x](x(w)) \\ &= x(yx(w)) + [y, x](x(w))\end{aligned}$$

which is in W. Moreover, $y(x^2(w))$ is a linear combination of w, $x(w)$ and $x^2(w)$. In general, for all $i \leqslant m-1$ and all $y \in I$, then $y(x^i(w)) \in W$ and $y(x^i(w))$ is a linear combination of w, $x(w)$, …, $x^i(w)$. And the coefficient of $x^i(w)$ in this linear combination is $\lambda(y)$ (Exercise).
So we see that for all $y \in I$ and for all $v \in W$, $y(v)$ is in W, so $y \colon W \to W$ is linear with fixed basis w, $x(w)$, …, $x^i(w)$. So we can construct a matrix of y with respect to the given basis:

$$[y|_W]_{w, x(w), \ldots, x^{m-1}(w)} = \begin{pmatrix} \lambda(y) & \lambda([y, x]) & \cdots & * \\ 0 & \lambda(y) & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda(y) \end{pmatrix}$$

which is upper triangular. Now take $z = [x, A] \in I$ (where $A \in I$), then $z \colon W \to W$ and

$$[z|_W]_{w, x(w), \ldots, x^{m-1}(w)} = \begin{pmatrix} \lambda(z) & \cdots & * \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda(z) \end{pmatrix}$$

so that $\operatorname{tr}([z|_W]) = \dim_F(W)\lambda(z) = m\lambda(z)$. But $z = [x, A] = xA - Ax \in \mathfrak{gl}(V)$; and since $A \colon W \to W$, $x \colon W \to W$, we get that $z|_W = (xA - Ax)|_W = x|_W \circ A|_W - A|_W \circ x|_W$. Hence $\operatorname{tr}(z|_W) = 0$, and so since $m \neq 0$, we must have $\lambda(z) = 0$ and so we are done! ■

Proposition IV.11. *Let L be a Lie subalgebra of $\mathfrak{gl}(V)$ such that for all $x \in L$, $x\colon V \to V$ is a nilpotent linear map. Then there exists $v \in V \setminus \{0\}$ such that $x(v) = 0$ for all $x \in L$.*

Proof. We proceed by induction on $\dim_F(L)$. When $\dim_F(L) = 1$, then $L = \langle y \rangle$ for some linear map $y\colon V \to V$. Since $y \in L$, y is nilpotent so for some $n \in \mathbb{N}$, $y^n = 0$ and $y^{n-1} \neq 0$ ($n \neq 0$ as L is 1-dimensional). Let $v = y^{n-1}(w)$. Then $y(v) = 0$, but for all $x \in L = \langle y \rangle$, $x = \alpha y$ for some $\alpha \in F$, hence $x(v) = \alpha y(v) = \alpha 0 = 0$.
Now we suppose that the proposition is true for all $K \subset \mathfrak{gl}(V)$ with $\dim_F(K) < \dim_F(L)$.
Let M be a maximal proper nontrivial subalgebra of L. Construct a map $\phi\colon M \to \mathfrak{gl}(L/M)$ such that for all $m \in M$, for all $y + M \in L/M$, $\phi(m)(y+M) = [m, y] + M$. Now is $\phi(m) \in \mathfrak{gl}(L/M)$? For $y_1 + M, y_2 + M \in L/M$:

$$\begin{aligned}\phi(m)((y_1 + M) + (y_2 + M)) = \phi(m)((y_1 + y_2) + M) &= [m, y_1 + y_2] + M \\ &= ([m, y_1] + [m, y_2]) + M \\ &= ([m, y_1] + M) + ([m, y_2] + M) \\ &= \phi(m)(y_1 + M) + \phi(m)(y_2 + M)\end{aligned}$$

For $\alpha \in F$, $y + M \in L/M$: $\phi(m)(\alpha(y + M)) = \alpha\phi(m)(y + M)$ (exercise).
So $\phi(m)$ is linear and hence in $\mathfrak{gl}(L/M)$.
Careful: $y + M = \widetilde{y} + M \Rightarrow \phi(m)(y + M) = \phi(m)(\widetilde{y} + M)$ (i.e. $\phi(m)$ is well-defined) (exercise).
What kind of map is ϕ? For $m_1, m_2 \in M$, for all $y + M \in L/M$:

$$\begin{aligned}\phi(m_1, m_2)(y + M) = [m_1 + m_2, y] + M &= ([m_1, y] + [m_2, y]) + M \\ &= ([m_1, y] + M) + ([m_2, y] + M) \\ &= \phi(m_1)(y + M) + \phi(m_1)(y + M)\end{aligned}$$

For $\alpha \in F$, $m \in M$, for all $y + M \in L/M$, $\phi(\alpha m)(y + M) = [\alpha m, y] + M = \alpha[m, y] + M = \alpha\phi(m)(y + M)$.
Hence ϕ is linear. Now for all $m_1, m_2 \in M$, and all $y + M \in L/M$:

$$\begin{aligned}[\phi(m_1), \phi(m_2)](y + M) &= ((\phi(m_1) \circ \phi(m_2)) - (\phi(m_2) \circ \phi(m_1)))(y + M) \\ &= ([m_1, [m_2, y]] - [m_2, [m_1, y]]) + M \\ &= ([m_1, [m_2, y]] + [m_2, [y, m_1]]) + M \\ &= -[y, [m_1, m_2]] + M = [[m_1, m_2], y]) + M \\ &= \phi([m_1, m_2])(y + M)\end{aligned}$$

Hence $\phi\colon M \to \mathfrak{gl}(L/M)$ is a Lie algebras homomorphism. For all $m \in M \subset L$, $m\colon V \to V$ is nilpotent, and hence m is ad-nilpotent. Thus there exists $k_m \in \mathbb{N}$ such that $\mathrm{ad}(m)^{k_m} = 0\colon L \to L$. Now $\phi(m)^{k_m} : y+M \mapsto [m, [m, \cdots [m, y]] \cdots] + M = \mathrm{ad}(m)^{k_m}(y) + M = M$. So for all $m \in M$, $\phi(m)$ is nilpotent. Now let's look at $\phi(M)$:

- $\phi(M)$ is a subalgebra of $\mathfrak{gl}(L/M)$
- $\forall \phi(m) \in \phi(M)$, $\phi(m)$ is nilpotent
- $\dim \phi(M) \leqslant \dim M < \dim L$

So by the induction hypothesis there exists $x_0 + M \in L/M \setminus \{0\}$ such that for all $\phi(m) \in \phi(M)$, $\phi(m)(x_0 + M) = 0_{L/M}$. Hence there exists $x_0 + M \in L/M \setminus \{0\}$ such that for all $m \in M$, $[m, x_0] \in M$.
Consider $L_0 := M \oplus \langle x_0 \rangle \subseteq L$. For all $x \in L_0$ there exists $m_x \in M$ and $\alpha_x \in F$ such that $x = m_x + \alpha_x x_0$. For all $x, \widetilde{x} \in L_0$, $[x, \widetilde{x}] = [m_x + \alpha_x x_0, \widetilde{m_x} + \widetilde{\alpha_x} x_0] \in M \subset L_0$, so L_0 is a subalgebra of L.
However since $L_0 \supset M$, by the maximality of M we must have $L_0 = L$. Notice that for all $m \in M$ and $x \in L$, $[m, x] = [m, m_x + \alpha_x x_0] \in M$ so M is an ideal of L. $\dim M = \dim L - 1 < \dim L$, $M \subset L \subseteq \mathfrak{gl}(V)$.
So by induction applied to M, there exists $x \in V \setminus \{0\}$ such that for all $m \in M$, $m(w) = 0$. Thus $W := \{v \in V \mid m(v) = 0 \ \forall m \in M\} \neq \{0\}$. Hence M has a weight $\lambda = 0$ with weight space $W = V_0$. So by IV.9, V_0 is L-invariant; in particular $x_0 \colon V_0 \to V_0$. But $x_0 \in L$ so x_0 is nilpotent and so there exists $g \in \mathbb{N}$ such that $x_0^g = 0 \colon V \to V$, hence $(x_0|_{V_0})^g = 0 \colon V_0 \to V_0$.
$\langle x_0|_{V_0} \rangle \subseteq \mathfrak{gl}(V_0)$, where $x_0|_{V_0}$ is nilpotent and $\dim\langle x_0 \rangle = 1$. Thus there exists $w_0 \in V_0 \setminus \{0\}$ such that $x_0(w_0) = 0$.
Now for all $x \in L$, $x = m_x + \alpha_x x_0$ and so $x(w_0) = m_x(w_0) + \alpha_x x_0(w_0) = 0$. ■

Engel's Theorem for Subalgebras of $\mathfrak{gl}(V)$. *Let L be a Lie subalgebra of $\mathfrak{gl}(V)$, with V a finite dimensional vector space over F. Suppose that for all $x \in L$, $x \colon V \to V$ is nilpotent. Then there exists a basis of V such that, with respect to the basis, all matrices of elements of L are strictly upper triangular; in particular, L is nilpotent.*

Proof. Suppose we have found a basis B of V such that for all $x \in L$, $[x]_B = \begin{pmatrix} 0 & \cdots & * \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$. Letting $n = \dim_F V$, we find that $\phi \colon \mathfrak{gl}(V) \to \mathfrak{gl}(n, F)$ defined by $\phi(T) = [T]_B$ is a Lie algebras isomorphism. We now have $\phi|_L \colon L \to u(n, F)$. $u(n, F)$ is nilpotent, hence so are $\phi^{-1}(u(n, F))$ and $L \subseteq \phi^{-1}(u(n, F))$.
We now have to show that we can find such a basis; we proceed by induction on $\dim V$. Suppose $\dim V = 1$, then $L \subseteq \mathfrak{gl}(V) \cong \mathfrak{gl}(1, F) \cong F$. For all $x \in L$, $x \colon V \to V$ is nilpotent. L is a subalgebra of $\mathfrak{gl}(V)$, hence $L = \{0\}$ or $L = \mathfrak{gl}(V)$ but $Id_V \notin L$ since Id_V is not nilpotent. Hence $L = \{0\}$ and there is nothing to prove.
If $\dim_F V > 1$, then by IV.11 there exists $v \in V \setminus \{0\}$ such that $\forall x \in L$, $x(v) = 0$. Consider $U = \langle v \rangle \leqslant V$, $\forall x \in L$ we may define $\bar{x} \colon V/U \to V/U$ such that for all $u + U \in V/U$, $\bar{x}(u + U) = x(u) + U$. We now have a correspondence $\phi \colon x \mapsto \bar{x}$, $\phi \colon L \to \mathfrak{gl}(V/U)$. (Have to check $\bar{x} \colon V/U \to V/U$ is linear, and that ϕ is well-defined.) In fact ϕ is a homomorphism (exercise). Then $\phi(L) \subseteq \mathfrak{gl}(V/U)$ and $\forall \bar{x} \in \phi(L)$, $\bar{x}$ is nilpotent (exercise).
But $\dim_F V/U < \dim_F V$ so by induction there is a basis $v_1 + U, \ldots, v_{n-1} + U$

of V/U such that with respect to this basis $[\bar{x}]$ is strictly upper triangular:

$$\begin{aligned}
x(v_1) &\in \bar{x}(v_1 + U) = 0_{V/U} = U \\
x(v_2) &\in \bar{x}(v_2 + U) = \alpha_1^2 v_1 + U \\
x(v_3) &\in \bar{x}(v_3 + U) = \alpha_2^3 v_2 + \alpha_1^3 v_1 + U \\
&\vdots \\
x(v_{n-1}) &\in \bar{x}(v_{n-1} + U) = \alpha_{n-2}^{n-1} v_{n-2} + \cdots + \alpha_1^{n-1} v_1 + U
\end{aligned}$$

Now we have to go back to V. $v_0 = v, v_1, v_2, \ldots, v_{n-1}$ is a basis for V (exercise) and we have:

$$\begin{aligned}
x(v_0) &= x(v) = 0 \\
x(v_1) &= \alpha_0^1 v_0 \\
x(v_2) &= \alpha_1^2 v_1 + \alpha_0^2 v_0 \\
&\vdots \\
x(v_{n-1}) &= \alpha_{n-2}^{n-1} v_{n-2} + \cdots + \alpha_0^{n-1} v_0
\end{aligned}$$

Hence

$$[x]_{v_0,\ldots,v_{n-1}} = \begin{pmatrix} 0 & \alpha_0^1 & \cdots & \alpha_0^{n-1} \\ 0 & 0 & \cdots & \alpha_1^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

which is strictly upper triangular. ■

Engel's Theorem. *If L is a Lie algebra such that $\forall x \in L$, x is* ad*-nilpotent, then L is a nilpotent Lie algebra.*

Proof. Consider $\mathrm{ad}\colon L \to \mathfrak{gl}(V)$ and look at $\mathrm{ad}(L)$. Since $\forall x \in L$, x is ad-nilpotent, we have $\forall z \in \mathrm{ad}(L)$, $z\colon L \to L$ is nilpotent, and $\mathrm{ad}(L) \subseteq \mathfrak{gl}(L)$. Thus by Engel's Theorem for subalgebras of $\mathfrak{gl}(V)$, $\mathrm{ad}(L)$ is a nilpotent Lie algebra. By the 1st Isomorphism Theorem $\mathrm{ad}(L) \cong L/\ker(\mathrm{ad}) = L/Z(L)$. Hence by IV.4, L is nilpotent. ■

Proposition IV.12. *Let V be a vector space over $\mathbb{C}$, $\dim_{\mathbb{C}} V < \infty$. Let L be a solvable Lie subalgebra of $\mathfrak{gl}(V)$. Then there exists $v \in V \setminus \{0\}$ such that v is a common eigenvector for all elements of L.*

Proof. We use induction on $\dim_{\mathbb{C}} L$. If $\dim_{\mathbb{C}} L = 1$ then $L = \langle y \rangle_{\mathbb{C}}$ for some $y\colon V \to V$. Since $\mathbb{C}$ is algebraically closed, y has an eigenvector $v \in V \setminus \{0\}$ and corresponding eigenvalue $\lambda \in \mathbb{C}$. Hence $\forall x \in L$, $x = \alpha y$ (some $\alpha \in \mathbb{C}$) and so $x(v) = \alpha y(v) = (\alpha\lambda)v$.
Now if $\dim_{\mathbb{C}} L > 1$, L is solvable and so $L \supset [L, L]$, look at $\bar{L} = L/[L, L]$ which is an abelian Lie algebra over $\mathbb{C}$. Since $\dim_{\mathbb{C}} L > \dim_{\mathbb{C}}[L, L]$, $\dim_{\mathbb{C}} \bar{L} \geqslant 1$. Hence we can find a subspace $\bar{M} \subset \bar{L}$ of codimension 1. Since $\bar{L}$ is abelian, $\bar{M}$ is an ideal of $\bar{L}$. By the Correspondence Theorem there is an ideal M of L such that $L \supset M \supseteq [L, L]$ with $\dim_{\mathbb{C}} M = \dim_{\mathbb{C}} L - 1$ as $L/M \cong (L/[L, L])/(M/[L, L])$ (by the 3rd Isomorphism Theorem).

We have $M \subseteq L \subseteq \mathfrak{gl}(V)$, M is solvable by III.7 and $\dim_{\mathbb{C}} M < \dim_{\mathbb{C}} L$ so by induction there exists $v \in V \setminus \{0\}$ such that $\forall m \in M$, $\exists \lambda(m) \in \mathbb{C}$ with $m(v) = \lambda(m)v$. But M is an ideal of L and we found a common eigenvector for all elements of M. Therefore there is a weight λ of M and a corresponding weight space $V_\lambda \ni v$ which is L-invariant by IV.10. Since $L \supset M$ take $z \in L \setminus M$ and look at $M \oplus \langle z \rangle \subseteq L$. But $\dim_{\mathbb{C}} M + \dim_{\mathbb{C}} \langle z \rangle = \dim_{\mathbb{C}} L$ so $M \oplus \langle z \rangle = L$. Now $z\colon V_\lambda \to V_\lambda$ has an eigenvector $w \in V_\lambda \setminus \{0\}$ with eigenvalue $\alpha_z \in \mathbb{C}$ since $\mathbb{C}$ is algebraically closed.
Now for all $x \in L$ there exists $m_x \in M$ and $a_x \in \mathbb{C}$ such that $x = m_x + a_x z$. Hence $x(w) = m_x(w) + a_x z(w) = \lambda(m_x)w + a_x \alpha_z w = (\lambda(m_x) + a_x \alpha_z)w$. So w is a common eigenvector for all elements of L. ■

Lie's Theorem. *Let V be a vector space over $\mathbb{C}$, $\dim_{\mathbb{C}} V = n$. Let L be a solvable subalgebra of $\mathfrak{gl}(V)$, then there exists a basis B of V such that for all $x \in L$, $[x]_B \in b(n, \mathbb{C})$ (i.e. $[x]_B$ is upper triangular).*

Proof. We proceed by induction on $\dim(V)$. The case $\dim(V) = 1$ is left as an exercise.
By the previous proposition, there exists a $v_0 \in V \setminus \{0\}$ and a $\lambda\colon L \to \mathbb{C}$ such that for all $x \in L$, $x(v_0) = \lambda(x)v_0$.
Take $U = \langle v_0 \rangle \subseteq V$. For all $x \in L$ and for all $u \in V$, we have that $x(u + v_0) = x(u) + x(v_0) \in X(u) + U$.
This allows us again to define $\phi\colon x \to \bar{x}$ where $\bar{x}$ is the map $V/U \to V/U$ given by $\bar{x}(u + U) = x(u) + U$.
Just as before, $\phi\colon L \to \mathfrak{gl}(V/U)$ is a Lie algebras homomorphism (exercise).
$\phi(L) \subseteq \mathfrak{gl}(V/U)$ and L is solvable, so $\phi(L)$ is solvable by Proposition III.7.
As $\dim(V/U) < \dim(V)$, we can proceed by induction; there is a basis $v_1 + U$, $v_2 + U, \ldots, v_{n-1} + U$ of V/U such that for all $\bar{x} \in \phi(L)$ we can write

$$[\bar{x}] = \begin{pmatrix} * & * & \cdots & * \\ 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & * \end{pmatrix}$$

so, expanding, we have

$$\begin{aligned} \bar{x}(v_1 + U) &= \alpha_1^1 v_1 + U \\ \bar{x}(v_2 + U) &= \alpha_2^2 v_2 + \alpha_1^2 v_1 + U \\ &\vdots \\ \bar{x}(v_{n-1} + U) &= \alpha_{n-1}^{n-1} v_{n-1} + \cdots + \alpha_1^{n-1} v_1 + U \end{aligned}$$

As for Engel's Theorem, it is easy to see that $v_0, v_1, \ldots, v_{n-1}$ is a basis of V (exercise). We then have

$$\begin{aligned} x(v_0) &= \lambda(x)v_0 = \alpha_0^0 v_0 \\ x(v_1) &= \alpha_1^1 v_1 + \alpha_0^1 v_0 \\ &\vdots \\ x(v_{n-1}) &= \alpha_{n-1}^{n-1} v_{n-1} + \cdots + \alpha_0^{n-1} v_0 \end{aligned}$$

and we can write

$$[x]_{v_0,v_1,\ldots,v_{n-1}} = \begin{pmatrix} \alpha_0^0 & \alpha_0^1 & \cdots & \alpha_0^{n-1} \\ 0 & \alpha_1^1 & \cdots & \alpha_1^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_{n-1}^{n-1} \end{pmatrix} \in \mathfrak{b}(n,\mathbb{C})$$

■

Corollary 1. *Let L be a solvable subalgebra of $\mathfrak{gl}(V)$, where V is a vector space over $\mathbb{C}$ of finite dimension. Then for all $x \in [L,L]$, $x\colon V \to V$ is nilpotent.*

Proof. $L \subseteq \mathfrak{gl}(V)$ so there exists a basis of V under which we can consider that $L \subseteq \mathfrak{b}(n,\mathbb{C})$.
But $[\mathfrak{b}(n,\mathbb{C}), \mathfrak{b}(n,\mathbb{C})] = \mathfrak{u}(n,\mathbb{C})$ so that $[L,L] \subseteq \mathfrak{u}(n,\mathbb{C})$. But $\mathfrak{u}(n,\mathbb{C})$ is nilpotent and hence so is $[L,L]$. ■

Corollary 2. *Let L be a Lie algebra over $\mathbb{C}$. Then*
L is solvable $\iff$ $[L,L]$ is nilpotent

Proof.

- $\Leftarrow$ $[L,L]$ is nilpotent implies that $[L,L]$ is solvable.
  On the other hand, $L/[L,L]$ is abelian and hence solvable. Therefore by Proposition III.7, L is solvable.

- $\Rightarrow$ L is solvable implies that $\mathrm{ad}(L) \subseteq \mathfrak{gl}(L)$ is solvable (Proposition III.7). But for all $x \in [\mathrm{ad}(L), \mathrm{ad}(L)] = \mathrm{ad}[L,L] \subseteq \mathfrak{gl}(V)$, x is nilpotent by Corollary 1. Hence $\mathrm{ad}_L([L,L])$ is nilpotent.
  By the First Isomorphism Theorem, we have that

  $$[L,L]/\left(Z(L) \cap [L,L]\right) \cong \mathrm{ad}_L([L,L])$$

  But $\mathrm{ad}_L([L,L])$ and $(Z(L) \cap [L,L]) \subseteq Z(L)$ are nilpotent, hence $[L,L]$ is nilpotent. ■

V Miscellaneous Topics

Definition V.1. Let L be a Lie algebra over F. A representation of L is a Lie algebras homomorphism

$$\phi\colon L \to \mathfrak{gl}(V)$$

where V is a vector space over F of finite dimension.
If $\ker(\phi) = \{0\}$, then ϕ is called a faithful representation.

Examples.

1. $L \subseteq \mathfrak{gl}(V)$. Then the incusion $i\colon L \to \mathfrak{gl}(V)$ is a faithful representation of L.

2. The trivial representation: $0\colon L \to \mathfrak{gl}(V)$ is defined by $0(x) = 0$ as a map $V \to V$ for all $x \in L$.

3. The adjoint representation: $\mathrm{ad}\colon L \to \mathfrak{gl}(L)$.

3a. An example of the adjoint representation, take $L = \mathfrak{sl}(2, \mathbb{C})$, $\mathrm{ad}\colon L \to \mathfrak{gl}(L)$.
Write $L = \langle \underbrace{e_{12}}_{e}, \underbrace{e_{11} - e_{22}}_{h}, \underbrace{e_{21}}_{f} \rangle_{\mathbb{C}}$.
Then $\mathrm{ad}(e)(e) = [e, e] = 0$, $\mathrm{ad}(e)(h) = [e, h] = -2e$, $\mathrm{ad}(e)(f) = [e, f] = h$, so we can write the matrix of $\mathrm{ad}(e)$ in the basis e, h, f:

$$[\mathrm{ad}(e)]_{e,h,f} = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Exercise: do the same for $\mathrm{ad}(h)$ and $\mathrm{ad}(f)$.

Definition V.2. Let L be a Lie algebra over F. A Lie module of L (or an L-module) is a vector space V over F of finite dimension equipped with an action $L \times V \to V$, written $(x, v) \mapsto xv$, that satisfies

$$(\alpha_1 x_1 + \alpha_2 x_2)v = \alpha_1(x_1 v) + \alpha_2(x_2 v)$$
$$x(\beta_1 v_1 + \beta_2 v_2) = \beta_1(x v_1) + \beta_2(x v_2)$$
$$[x, y]v = x(yv) - y(xv)$$

for all $x_1, x_2, x, y \in L$, all $v, v_1, v_2 \in V$ and all $\alpha_1, \alpha_2, \beta_1, \beta_2 \in F$.

Example. Take $L \subseteq \mathfrak{gl}(V)$. Then V is an L-module if we define $xv = x(v)$.

Lemma V.3. *Definition V.1 is equivalent to Definition V.2.*
That is, if ϕ is a representation of L, $\phi\colon L \to \mathfrak{gl}(V)$, then V is an L-module under the action $xv = \phi(x)(v)$.
And if V is an L-module, then we get a representation $\phi\colon L \to \mathfrak{gl}(V)$ by defining $\phi(x)(v) = xv$.

Proof. Exercise. ∎

VI The Killing Form

Let V be a vector space over $\mathbb{C}$ with $\dim_{\mathbb{C}} V < \infty$, $x\colon V \to V$ be linear. From Algebra I there is a basis B of V such that

$$[x]_B = \begin{pmatrix} P_1 & & & 0 \\ & P_2 & & \\ & & \ddots & \\ 0 & & & P_k \end{pmatrix} \quad \text{where} \quad P_i = \begin{pmatrix} \lambda_i & 1 & & 0 \\ 0 & \lambda_i & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{pmatrix}$$

$$[x]_B = \underbrace{\begin{pmatrix} \alpha_1 & & & 0 \\ & \alpha_2 & & \\ & & \ddots & \\ 0 & & & \alpha_l \end{pmatrix}}_{D} + \underbrace{\begin{pmatrix} 0 & 1 & & 0 \\ 0 & 0 & \ddots & \\ & & \ddots & 0 \\ 0 & & & 0 \end{pmatrix}}_{N}.$$

So there are linear maps $d\colon V \to V$, $n\colon V \to V$ such that $[d]_B = D$, $[n]_B = N$ and hence $x = d + n$, where d is diagonalisable and n is nilpotent. We also see that $d \circ n = n \circ d$ since $DN = ND$. Under these conditions, $x = d+n$ is called a Jordan decomposition of x. d is called the semisimple part, and n the nilpotent part. If $x = d$ is diagonalisable then we will say that x is semisimple.

The next lemma is from algebra and will not be proved here.

Lemma VI.1. *Let $d\colon V \to V$ be a diagonalisable linear map s.t. $x\colon V \to V$ has Jordan decompostion $x = d + n$. Then:*

1. *There is a polynomial $p(t) \in \mathbb{C}[t]$ such that $p(x) = d$*
2. *If B is a basis of V such that $[d]_B = D$ which is diagonal, then there is a polynomial $q(t) \in \mathbb{C}[t]$ such that $q(x) = \bar{d}$ where $[\bar{d}]_B = \bar{D}$.*

Lemma VI.2. *Let $x \in \mathfrak{gl}(V)$, where V is a vector space over $\mathbb{C}$, $\dim_{\mathbb{C}} V < \infty$. Suppose x has a Jordan decompostion $x = d + n$. Consider $\operatorname{ad}(x) \in \mathfrak{gl}(\mathfrak{gl}(V))$, $\operatorname{ad}(x)$ has Jordan decomposition $\operatorname{ad}(x) = \operatorname{ad}(d) + \operatorname{ad}(n)$.*

Proof. For all $y \in \mathfrak{gl}(V)$, $\operatorname{ad}(x)(y) = [x, y] = [d + n, y] = [d, y] + [n, y] = \operatorname{ad}(d)(y) + \operatorname{ad}(n)(y)$. Thus $\operatorname{ad}(x) = \operatorname{ad}(d) + \operatorname{ad}(n)$.
$n\colon V \to V$ is nilpotent $\Rightarrow$ n is ad-nilpotent $\Rightarrow$ $\operatorname{ad}(n)$ is nilpotent.
We can choose a basis B of V such that $[d]_B = \begin{pmatrix} \alpha_1 & & 0 \\ & \ddots & \\ 0 & & \alpha_n \end{pmatrix}$. Now $\mathfrak{gl}(V) \cong \mathfrak{gl}(n, \mathbb{C})$. Hence we can consider $\operatorname{ad}(d)\colon \mathfrak{gl}(n, \mathbb{C}) \to \mathfrak{gl}(n, \mathbb{C})$. Look at $\{e_{ij}\}_{i,j=1}^n$. $\operatorname{ad}(d)(e_{ij}) = [d, e_{ij}] = de_{ij} - e_{ij}d = (\alpha_i - \alpha_j)e_{ij}$. So e_{ij} is an eigenvector of $\operatorname{ad}(d)$ and so $\{e_{ij}\}_{i,j=1}^n$ is a basis of eigenvectors of $\operatorname{ad}(d)$. Therefore $\operatorname{ad}(d)$ is diagonalisable.
It remains to show that $\operatorname{ad}(d)$ and $\operatorname{ad}(n)$ commute. But $\operatorname{ad}(d) \circ \operatorname{ad}(n) - \operatorname{ad}(n) \circ \operatorname{ad}(d) = [\operatorname{ad}(d), \operatorname{ad}(n)] = \operatorname{ad}([d, n]) = \operatorname{ad}(0) = 0$. ■

Proposition VI.3. *Let V be a vector space over $\mathbb{C}$, $\dim_{\mathbb{C}} V < \infty$. Let L be a Lie subalgebra of $\mathfrak{gl}(V)$. If for all $x \in L$, $y \in [L, L]$, $\operatorname{tr}(x \circ y) = 0$, then L is solvable.*

Proof. Let's show that for all $x \in [L, L]$, x is nilpotent. Then by Engel's Theorem, $[L, L]$ will be nilpotent and so L will be solvable.

Let $x \in [L, L] \subseteq L \subseteq \mathfrak{gl}(V)$. x has Jordan decomposition $x = d + n$, now x is nilpotent iff $d = 0$. Choose a basis B such that $[x]_B$ is in JCF, hence $[d]_B = \begin{pmatrix} \alpha_1 & & 0 \\ & \ddots & \\ 0 & & \alpha_n \end{pmatrix}$. So $d = 0 \iff |\alpha_1|^2 + |\alpha_2|^2 + \cdots + |\alpha_n|^2 = 0 \iff \sum_{i=1}^n \alpha_i \bar{\alpha_i} = 0$. Also $\mathrm{tr}(\bar{d} \circ x) = \mathrm{tr}([\bar{d}]_B [x]_B) = \mathrm{tr}\left(\begin{pmatrix} \bar{\alpha_1} & & 0 \\ & \ddots & \\ 0 & & \bar{\alpha_n} \end{pmatrix} \begin{pmatrix} \alpha_1 & & * \\ & \ddots & \\ 0 & & \alpha_n \end{pmatrix}\right) = \sum_{i=1}^n \alpha_i \bar{\alpha_i}$.

We now have x is nilpotent iff $\mathrm{tr}(\bar{d} \circ x) = 0$. Since $x \in [L, L]$, there exist $y_1, \ldots, y_k, z_1, \ldots, z_k \in L$ such that $x = [y_1, z_1] + \cdots + [y_k, z_k]$. We are looking at $\mathrm{tr}(\bar{d} \circ x) = \mathrm{tr}(\bar{d} \circ \sum_{i=1}^k [y_i, z_i]) = \mathrm{tr}(\sum_{i=1}^k \bar{d} \circ [y_i, z_i]) = \sum_{i=1}^k \mathrm{tr}(\bar{d} \circ [y_i, z_i])$. If we can show that $\mathrm{tr}(\bar{d} \circ [y, z]) = 0$ for all $y, z \in L$ then we are done.

Important equality: $\mathrm{tr}(A \circ [B, C]) = \mathrm{tr}([A, B] \circ C)$ for all $A, B, C \in \mathfrak{gl}(V)$. (Check!)

Thus $\mathrm{tr}(\bar{d} \circ [y, z]) = \mathrm{tr}([\bar{d}, y] \circ z)$. If we can show that $[\bar{d}, y] \in L$ then $\mathrm{tr}([\bar{d}, y]) = 0$ and we are done. $[\bar{d}, y] = \mathrm{ad}(\bar{d})(y) \in L$ for all $y \in L$ iff $\mathrm{ad}(\bar{d})\colon L \to L$. Since $x = d + n$ is a Jordan decomposition, $\mathrm{ad}(x) = \mathrm{ad}(d) + \mathrm{ad}(n)$ is a Jordan decomposition. Thus part 2. of VI.1 gives a polynomial $q(t) \in \mathbb{C}[t]$ such that $q(\mathrm{ad}(x)) = \overline{\mathrm{ad}(d)} = \mathrm{ad}(\bar{d})$. Hence $\mathrm{ad}(\bar{d})(y) = q(\mathrm{ad}(x)(y)) \in L$. ∎

Corollary VI.4. *Let L be a Lie algebra over $\mathbb{C}$, then L is solvable iff for all $x \in L$ and $y \in [L, L]$, $\mathrm{tr}(\mathrm{ad}(x) \circ \mathrm{ad}(y)) = 0$.*

Proof.

- ⇐ The propostion tells us that $\mathrm{ad}(L) \subseteq \mathfrak{gl}(L)$ is solvable. But then by the 1st Isomorphism Theorem, $\mathrm{ad}(L) \cong L/Z(L)$ and so L is also solvable.
- ⇒ Since L is solvable, so is $\mathrm{ad}(L)$ which is a subalgebra of $\mathfrak{gl}(L)$. Hence (by Lie's Theorem) there is a basis B of L such that for all $x \in L$, $[\mathrm{ad}(x)]_B \in b(n, \mathbb{C})$. Hence for all $x, y \in L$, $[\mathrm{ad}([x, y])]_B \in u(n, \mathbb{C})$. This gives for $x \in L$ and $y \in [L, L]$, $\mathrm{tr}(\mathrm{ad}(x) \circ \mathrm{ad}(y)) = \mathrm{tr}(\underbrace{[\mathrm{ad}(x)]_B}_{\in b(n,\mathbb{C})} \underbrace{[\mathrm{ad}(y)]_B}_{\in u(n,\mathbb{C})}) = \mathrm{tr}(\underbrace{*}_{\in u(n,\mathbb{C})}) = 0$.

∎

Definition VI.5. Let L ba a Lie algebra over $\mathbb{C}$. The *Killing form* on L is $k\colon L \times L \to \mathbb{C}$ defined by $k(x, y) = \mathrm{tr}(\mathrm{ad}(x) \circ \mathrm{ad}(y))$ for all $x, y \in L$.

k is bilinear since tr and ad are linear. k is symmetric since $\mathrm{tr}(AB) = \mathrm{tr}(BA)$. Also $k([x, y], z) = k(x, [y, z])$ (check this), called the associativity of k. We can now restate VI.4 as:

Cartan's 1st Criterion. *Let L be a Lie algebra over $\mathbb{C}$, then L is solvable iff $k(x, y) = 0$ for all $x \in L$ and $y \in [L, L]$.*

Review of Bilinear Forms

Let V be a vector space over $\mathbb{C}$, $\dim_{\mathbb{C}} V < \infty$. Let $\beta\colon V \times V \to \mathbb{C}$ be a symmetric bilinear form. For any subspace $A \subseteq V$ define $A^{\perp} := \{v \in V \mid \beta(v,w) = 0 \ \forall w \in A\}$.

Definition. β is *non-degenerate* if $V^{\perp} = \{0\}$.

Definition. $V^* := \{\alpha\colon V \to \mathbb{C} \mid \alpha \text{ is linear}\}$ is the *dual* to V.

$\dim V^* = \dim V$.

Lemma VI.6. *If β is non-degenerate, and W is a subspace of V, then* $\dim W + \dim W^{\perp} = \dim V$.

Proof. Consider $f\colon V \to V^*$ defined by $f(v)(w) = \beta(v,w)$ for all $v, w \in V$. Exercise: check $f\colon V \to V^*$ (i.e. $f(v)$ is linear); now check that f is linear. Now

$$\begin{aligned}\ker f &= \{v \in V \mid f(v) = 0_{V^*}\} = \{v \in V \mid \beta(v,w) = 0 \ \forall w \in V\}\\ &= V^{\perp} = \{0\}\end{aligned}$$

since β is non-degenerate. Therefore f is injective and hence an isomorphism. Consider $r\colon V^* \to V^*$ defined by $r(\alpha) = \alpha|_W$. r is linear and surjective. Let $w_1, \dots, w_k$ be a basis of W, and extend it to a basis $w_1, \dots, w_k, v_{k+1}, \dots, v_n$ of V. For all $\alpha \in X^*$ by MA106 there is a unique linear map $\tilde{\alpha}\colon V \to \mathbb{C}$ with $\tilde{\alpha}(w_i) = \alpha(w_i)$ and $\tilde{\alpha}(v_i) = 0$.

$$\begin{aligned}\ker r &= \{\alpha \in V^* \mid \alpha|_W = 0_{W^*}\} = \{\alpha \in V^* \mid \alpha(w) = 0 \ \forall w \in W\}\\ &= \{f(v) \mid f(v)(w) = 0 \ \forall w \in W\} = \{f(v) \mid \beta(v,w) = 0 \ \forall w \in W\}\\ &\cong \{v \in V \mid \beta(v,w) = 0 \ \forall w \in W\} = W^{\perp}\end{aligned}$$

Finally $\dim V = \dim V* = \dim \ker r + \dim \operatorname{im} r = \dim W^{\perp} + \dim W$. ∎

Corollary. $W \cap W^{\perp} = \{0\} \Rightarrow V = W \oplus W^{\perp}$.

Let β be a symmetric bilinear form on V, $e_1, \dots, e_n$ a basis of V. Define the matrix of β to be $(\beta(e_i, e_j))_{n \times n}$.

Theorem. *β is non-degenerate iff* $\det((\beta(e_i,e_j))_{n\times n}) \neq 0$.

Lemma VI.7. *Let I be an ideal of L, then we can look at it as a Lie algebra in its own right. Let k_I be the Killing form of I, k the Killing form of L. Then for all $x, y \in L$, $k(x,y) = k_I(x,y)$.*

Proof. Let B_I be a basis of I and extend it to a basis B of L. For all $x \in I$ and $y \in L$, $[x,y] \in I$, thus $\operatorname{ad}(x)\colon L \to I$. Thus $[\operatorname{ad}(x)]_B = \left(\begin{array}{c|c} A_x & B_x \\ 0 & 0 \end{array}\right)$, where $A_x = [\operatorname{ad}(x)|_I]_{B_I}$. Similarly for $z \in I$, $[\operatorname{ad}(x)]_B = \left(\begin{array}{c|c} A_z & B_z \\ 0 & 0 \end{array}\right)$, where $A_z = [\operatorname{ad}(z)|_I]_{B_I}$. Now

$$\begin{aligned}k(x,z) &= \operatorname{tr}([\operatorname{ad}(x)]_B[\operatorname{ad}(z)]_B) = \operatorname{tr}\left(\begin{array}{c|c} A_xA_z & * \\ 0 & 0 \end{array}\right)\\ &= \operatorname{tr}(A_xA_z) = \operatorname{tr}([\operatorname{ad}(x)|_I]_{B_I}[\operatorname{ad}(z)|_I]_{B_I})\\ &= k_I(x,z)\end{aligned}$$

∎

Lemma VI.8. *If I is an ideal of L, then so is $I^\perp$.*

Proof. $I^\perp = \{x \in L \mid k(x,y) = 0\ \forall i \in I\}$, clearly this is a subspace of L (since k is bilinear). For all $x \in L$ and $y \in I^\perp$ we want $[y,x] \in I^\perp$. For all $i \in I$, $k([y,x],i) = k(y,[x,i]) = 0$ and so $[y,x] \in I^\perp$. ■

Cartan's 2nd Criterion. *L is semisimple iff k is non-degenerate. (i.e.* $\mathrm{Rad}(L) = \{0\} \iff L^\perp = \{0\}$*.)*

Proof. $L^\perp$ is an ideal of L, and for all $x \in L^\perp$ and $y \in [L^\perp, L^\perp] \subseteq L$, $k(x,y) = 0 = k_{L^\perp}(x,y)$. Thus $L^\perp$ is solvable. Therefore $L^\perp$ is a solvable ideal of L, so $L^\perp \subseteq \mathrm{Rad}(L)$.

- $\Rightarrow$ L is semisimple $\Rightarrow L^\perp \subseteq \mathrm{Rad}(L) = \{0\} \Rightarrow k$ is non-degenerate.
- $\Leftarrow$ Assume $\mathrm{Rad}(L) \neq \{0\}$, claim that there exists $I_0 \neq \{0\}$ which is an ideal of L, abelian, and $I_0 \subseteq \mathrm{Rad}(L)$. The proof is an exercise (hint: $I_0 = \mathrm{Rad}(L)^{(m)}$ where $\mathrm{Rad}(L)^{(m+1)} = \{0\} \neq \mathrm{Rad}(L)^{(m)}$).
  For all $x, y \in L$ and $i \in I_0$, $\mathrm{ad}(i) \circ \mathrm{ad}(x) \circ \mathrm{ad}(i)(y) = [\underbrace{i}_{\in I_0}, \underbrace{[x,[i,y]]}_{\in I_0}] = 0$.
  Thus $\mathrm{ad}(x) \circ \mathrm{ad}(i) \circ \mathrm{ad}(x) \circ \mathrm{ad}(i) = 0$. Hence $\mathrm{ad}(x) \circ \mathrm{ad}(i)$ is a nilpotent linear map (in $\mathfrak{gl}(L)$). Hence $\langle \mathrm{ad}(x) \circ \mathrm{ad}(i) \rangle$ is nilpotent. Thus by Engel's Theorem for Subalgebras of $\mathfrak{gl}(L)$ there is a basis B such that the matrix of $\mathrm{ad}(x) \circ \mathrm{ad}(i)$ is strictly upper triangular. Hence $k(x,i) = \mathrm{tr}(\mathrm{ad}(x) \circ \mathrm{ad}(i)) = 0$. Now for all $x \in L$, and all $i \in I_0$, $k(x,i) = 0$. Hence $I_o \subseteq L^\perp$, so $L^\perp \neq 0$, so k is degenerate. ■

Example. $L = \mathfrak{sl}(2,\mathbb{C})$, k the Killing form. Let's calculate the matrix of k. A basis of $\mathfrak{sl}(2,\mathbb{C})$ is $e = e_{12}, h = e_{11} - e_{22}, f = e_{21}$. With respect to this basis:

$$[\mathrm{ad}(e)] = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad [\mathrm{ad}(h)] = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad [\mathrm{ad}(f)] = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}$$

$$[k] = \begin{pmatrix} k(e,e) & k(h,e) & k(f,e) \\ k(e,h) & k(h,h) & k(f,h) \\ k(e,f) & k(h,f) & k(f,f) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 4 \\ 0 & 8 & 0 \\ 4 & 0 & 0 \end{pmatrix}$$

$\det([k]) \neq 0$ so k is non-degenerate. Hence $\mathfrak{sl}(2,\mathbb{C})$ is semisimple.
Notice that $k(e,e) = 0$, $\langle e \rangle \subseteq \langle e \rangle^\perp = \langle e, h \rangle$.

Lemma VI.9. *Let L be a semisimple Lie algebra over $\mathbb{C}$. If $I \neq \{0\}$ is an ideal of L, then so is $I^\perp$, and $L = I \oplus I^\perp$. Moreover, I is semisimple.*

Proof. By VI.8, $I^\perp$ is an ideal of L, so $I \cap I^\perp$ is an ideal of L. For all $x, y \in I \cap I^\perp$, $k(x,y) = 0$. But $k(x,y) = k_{I \cap I^\perp}(x,y) = 0$. So by Cartan's 1st Criterion, $I \cap I^\perp$ is solvable. Thus $I \cap I^\perp \subseteq \mathrm{Rad}(L) = \{0\}$. Hence $L = I \oplus I^\perp$.
I is semisimple $\iff k_I$ is non-degenerate $\iff \{j \in I \mid k_I(j,i) = 0\ \forall i \in I\} = \{0\}$. But $\{j \in I \mid k_I(j,i) = 0\ \forall i \in I\} = \{j \in I \mid k(j,i) = 0\ \forall i \in I\} = I \cap I^\perp = \{0\}$ ■

Exercise. If I is an ideal, $(I^\perp)^\perp = I$.

Corollary VI.10. *If L is a semisimple Lie algebra over $\mathbb{C}$, and I is an ideal of L, then L/I is semisimple.*

Proof. By VI.9, $L = I \oplus I^\perp$ and $I, I^\perp$ are both semisimple. Now $L/I = (I \oplus I^\perp)/I = (I + I^\perp)/I \cong I^\perp/(I \cap I^\perp) \cong I^\perp$ is semisimple. ■

Corollary VI.11. *If L is a semisimple Lie algebra over $\mathbb{C}$, then every homomorphic image of L is semisimple.*

Proof. Let $\phi\colon L \to M$ be a homomorphism of Lie algebras. Then $\phi(L) \cong L/\ker\phi$, but $\ker\phi$ is an ideal of L. So $\phi(L)$ is semisimple by VI.10. ■

Definition VI.12. Let M be a Lie algebra over $\mathbb{C}$. Then M is *simple* if M is non-abelian and has no proper ideals.

Theorem VI.13. *If L is a Lie algebra over $\mathbb{C}$, then L is semisimple iff $L = L_1 \oplus \cdots \oplus L_k$ where $\forall i$, L_i is a simple ideal of L.*

Proof.

- $\Rightarrow$ Suppose L is semisimple. Let I be a minimal proper ideal of L. That is, $\{0\} \neq I \neq L$, I an ideal of L, and if J is an ideal of L with $J \subseteq I$ then $J = \{0\}$ or $J = I$.
  By VI.9, $L = I \oplus I^\perp$. If I_0 is an ideal of I, then for all $x \in L$: $x = i + j$ for some $i\, inI$, $j \in I^\perp$. Now for all $x \in I_0$, $[a, x] = [a, i + j] = [a, i] + [a, j] = [a, i] \in I_0$, since $[I, I^\perp] \subseteq I \cap I^\perp = \{0\}$.
  Hence I_0 is an ideal of L, so $I_0 = \{0\}$ or $I_0 = I$. Therefore I has no proper ideals. I is also non-abelian since it cannot be solvable since L is semisimple. Hence I is simple.
  If $I^\perp \neq \{0\}$, we can use induction on the dimension (since $\dim I^\perp < \dim L$ and $I^\perp$ is semisimple) to decompose $I^\perp = L_2 \oplus \cdots \oplus L_k$ where each L_i is a simple ideal of $I^\perp$. Now $L = I \oplus L_2 \oplus \cdots \oplus L_k$. It remains to show that each L_i is an ideal of L.
  For all $x \in L$, $x = i + a$ for $i \in I$, $a \in I^\perp$. Then for $z \in L_i$, $[x, z] = [i, z] + [a, z] \in L_i$. Hence L_i is an ideal of L. We let $L_1 := I$ and hence $L = L_1 \oplus \cdots \oplus L_k$.

- $\Leftarrow$ $L = L_1 \oplus \cdots \oplus L_k$. Let $R = \mathrm{Rad}(L)$, consider $[R, L_i]$. $[R, L_i] \subseteq R$ so $[R, L_i]$ is a solvable ideal of L. $[R, L_i] \subseteq L_i$ so $[R, L_i]$ is a solvable ideal of L_i. Hence $[R, L_i] = \{0\}$ or $[R, L_i] = L_i$ since L_i is simple. If $[R, L_i] = L_i$ then L_i is solvable and hence L_i is abelian, which is a contradiction.
  Therefore $[R, L_i] = \{0\}$. Now for all $x \in L$, $x = a_1 + a_2 + \cdots a_k$ with $a_i \in L_i$. For all $r \in R$, $[r, x] = [r, a_1] + \cdots + [r, a_k] = 0$. Thus $R \subseteq Z(L)$. But $L = L_1 \oplus \cdots \oplus L_k$ and so $Z(L) = Z(L_1) \oplus \cdots \oplus Z(L_n)$. But all of the L_i are simple, hence $Z(L_i) = \{0\}$. Hence $Z(L) = \{0\}$ and therefore $R = \{0\}$. Thus L is semisimple. ■

We remark that simple $\Rightarrow$ semisimple. Now to study semisimple Lie algebras we have to find out about simple Lie algebras.

In section II we looked at $\mathrm{ad}\colon L \to \mathfrak{gl}(L)$, in fact $\mathrm{ad}(L) \subseteq \mathrm{Der}(L) \subseteq \mathfrak{gl}(L)$.

Theorem VI.14. *If L is a semisimple Lie algebra over $\mathbb{C}$, then $L \cong \mathrm{ad}(L) = \mathrm{Der}(L)$. (i.e. all derivations are inner.)*

Proof. Consider $\mathrm{ad}(L) \cong L/Z(L)$. As L is semisimple, $Z(L) = \{0\}$, so $\mathrm{ad}(L) \cong L$. Denote $M := \mathrm{ad}(L)$. $M \subseteq \mathrm{Der}(L) \subseteq \mathfrak{gl}(L)$, M is semisimple. Let k_M be

the Killing form of M. By Cartan's 2nd Criterion, k_M is non-degenerate. We claim that for all $x \in L$ and all $\delta \in \mathrm{Der}(L)$, $[\delta, \mathrm{ad}(x)] = \mathrm{ad}(\delta(x))$. The proof is an exercise.
$[\delta, \mathrm{ad}(x)] = \mathrm{ad}(\delta(x)) \in M$, so $[\mathrm{Der}(L), M] \subseteq M$. Hence M is an ideal of $\mathrm{Der}(L)$. Let $k_{\mathrm{Der}(L)}$ be the Killing form of $\mathrm{Der}(L)$. $k_{\mathrm{Der}(L)}|_M = k_M$. Look at $M^\perp = \{\delta \in \mathrm{Der}(L) \mid k_{\mathrm{Der}(L)}(\delta, m) = 0\ \forall m \in M\}$. Now $M \cap M^\perp = \{\delta \in M \mid k_M(\delta, m) = 0\ \forall m \in M\} = \{0\}$ since M is semisimple. By the "small review", $\mathrm{Der}(L) = M \oplus M^\perp$ and $[M, M^\perp] \subseteq M \cap M^\perp = \{0\}$. Now for all $\delta \in M^\perp$ and $x \in L$, $\mathrm{ad}(\delta(x)) = [\delta, \mathrm{ad}(x)] = \{0\}$, and so $\delta(x) \in \ker(\mathrm{ad}) = Z(L) = \{0\}$, and so $\delta(x) = 0$. Hence $\delta = 0$. Therefore $M^\perp = \{0\}$. Therefore $M = \mathrm{Der}(L)$. ■

Lemma VI.15. *If L is a Lie algebra over $\mathbb{C}$ and $\delta \in \mathrm{Der}(L)$, and $\delta = d_\delta + \eta_\delta$ is a Jordan decomposition of $\delta \in \mathfrak{gl}(L)$, then both d_δ, η_δ are derivations of L.*

Proof. If we can show that $d_\delta \in \mathrm{Der}(L)$ then we will have $\eta_\delta = \delta - d_\delta \in \mathrm{Der}(L)$. For all $\lambda \in \mathbb{C}$, let $L_\lambda = \{x \in L \mid (\delta - \lambda I_L)^i(x) = 0 \text{ for some } i = i(x) \in \mathbb{N}\}$ be the generalised eigenspace of λ. If λ is an eigenvalue of δ, then $L_\lambda \neq \{0\}$, otherwise $L_\lambda = \{0\}$. λ is an eigenvalue of δ iff λ is an eigenvalue of d_δ. Each L_λ is a subspace of L, in fact $L = \bigoplus_{\lambda \text{ is an eigenvalue of } \delta} L_\lambda$ (quoted without proof).
Exercise: show that $[L_\alpha, L_\beta] \subseteq L_{\alpha+\beta}$ (hint: show that $(\delta - (\alpha+\beta)I_L)^k([x, y]) = \sum_{i=0}^{k} [(\delta - \alpha I_L)^i(x), (\delta - \beta I_L)^{k-i}(y)]$ by induction on k).
Now for all $a \in L_\alpha$, $b \in L_\beta$, we have $d_\delta([a, b]) = (\alpha+\beta)[a, b]$, and also $[a, d_\delta(b)] + [d_\delta(a), b] = [a, \beta b] + [\alpha a, b] = (\beta + \alpha)[a, b]$. Hence $d_\delta \in \mathrm{Der}(L)$. ■

Corollary VI.16. *Let L be a semisimple Lie algebra over $\mathbb{C}$, then for all $x \in L$, there exists a $d, n \in L$ with $x = d + n$, $[d, n] = 0$, $\mathrm{ad}(d)$ diagonalisable and $\mathrm{ad}(n)$ nilpotent.*

Proof. Look at $\mathrm{ad}(x) \in \mathfrak{gl}(L)$ which has Jordan decomposition $\mathrm{ad}(x) = \delta + \eta$ with δ diagonalisable, η nilpotent and $[\delta, \eta] = 0$. But $\mathrm{ad}(x) \in \mathrm{Der}(L)$ so $\delta, \eta \in \mathrm{Der}(L)$. But L is semisimple and hence $\mathrm{ad}(L) = \mathrm{Der}(L)$. Now there exists $d, n \in L$ such that $\mathrm{ad}(d) = \delta$ and $\mathrm{ad}(n) = \eta$. Hence $x = d + n$ and $0 = [\delta, \eta] = [\mathrm{ad}(d), \mathrm{ad}(n)] = \mathrm{ad}([d, n])$ which gives $[d, n] \in \ker(\mathrm{ad}) = Z(L) = \{0\}$. ■

Definition VI.17. Let L be a semisimple Lie algebra over $\mathbb{C}$. Then for all $x \in L$ let d, n be as above. The decomposition $x = d + n$ is called an *abstract Jordan decomposition.* d is called the *semisimple part* of x and n the *nilpotent part* of x. If $d = 0$, x is *nilpotent* and if $n = 0$ x is *semisimple.*

Remark. Notice that if $x \in L \subseteq \mathfrak{gl}(V)$, then x has a Jordan decomposition and also an abstract Jordan decomposition. Are they connected? In fact, they coincide.

This theorem is too long to prove here, but we will use it for the remark which follows:

Theorem VI.18. *Let L be a semisimple Lie algebra over $\mathbb{C}$ and $\phi\colon L \to \mathfrak{gl}(V)$ be a representation of L. Now if $x \in L$ and $x = d + n$ is an abstract Jordan decomposition, then $\phi(x) = \phi(d) + \phi(n)$ is a Jordan decomposition of $\phi(x) \in \mathfrak{gl}(V)$.*

Remark. Let L be a semisimple Lie algebra over $\mathbb{C}$, $x \in L$ and $x = d + n$ be an abstract Jordan decomposition. Then $\mathrm{ad}(x) = \mathrm{ad}(d) + \mathrm{ad}(n)$ is a Jordan decomposition. Suppose $y \in L$ and $[x, y] = 0$. Then $[d, y] = [n, y] = 0$.

Proof. Since $[n, y] = [x - d, y]$ it is enough to show that $[d, y] = \mathrm{ad}(d)(y) = 0$. As $x = d + n$ is an abstract Jordan decomposition, $\mathrm{ad}(d)$ diagonalisable and $\mathrm{ad}(x) = \mathrm{ad}(d) + \mathrm{ad}(n)$ is a Jordan decomposition of $\mathrm{ad}(x)$. By VI.1 there is a polynomial $p(t) \in \mathbb{C}[t]$ with $p(\mathrm{ad}(x)) = \mathrm{ad}(d)$. Hence $\mathrm{ad}(d)(y) = p(\mathrm{ad}(x))(y) = 0$. ■

VII Cartan Subalgebras

Let L be a nontrivial semisimple Lie algebra over $\mathbb{C}$.

Definition VII.1. Let H be a subalgebra of L such that:

1. H is abelian.
2. $\forall h \in H$, h is semisimple.
3. H is maximal with respect to 1. and 2.

Then H is called a *Cartan subalgebra* of L.

Lemma VII.2. *L contains a nontrivial Cartan subalgebra.*

Proof. For all $x \in L$ we have an abstract Jordan decomposition $x = d + n$, and so $\mathrm{ad}(x) = \mathrm{ad}(d) + \mathrm{ad}(n)$. If for all $x \in L$, $\mathrm{ad}(d) = 0$ then $\mathrm{ad}(x) = \mathrm{ad}(n) \in \mathfrak{gl}(L)$ is nilpotent. Therefore by Engel's theorem L is nilpotent and we have a contradiction since L is semisimple.
So there is some $x \in L$ with abstract Jordan decomposition $x = d+n$ with $d \neq 0$, $d \in L$ and d semisimple. Now we take $\langle d \rangle_{\mathbb{C}}$ which is an abelian subalgebra of L and for all $y \in \langle d \rangle_{\mathbb{C}}$, y is semisimple. Now any maximal element of $S := \{A \mid A$ is a subalgebra of L, A abelian and $\forall a \in A$, a is semisimple$\} \neq \emptyset$ is a Cartan subalgebra of L. ■

Notation. *For $y \in L$, $C_L(y) := \{x \in L \mid [x, y] = 0\}$.*
For $A \subseteq L$, $C_L(A) := \{x \in L \mid [x, a] = 0 \ \forall a \in A\}$.

Lemma VII.3. *Let H be a Cartan subalgebra of L. Then $C_L(H)$ is a subalgebra of L, and for all $h \in H$, $C_L(h)$ is a subalgebra of L, and $H \subseteq C_L(H) \subseteq C_L(h)$.*

Proof. H is abelian, so $H \subseteq C_L(H)$ and $h \in H$, so $C_L(H) \subseteq C_L(h)$. $C_L(H)$ and $C_L(h)$ are vector spaces by their definitions and bilinearity of $[\cdot,\cdot]$. The rest is an exercise. ■

Lemma VII.4. *Let $h_0 \in H$, with $\dim_{\mathbb{C}} C_L(h_0) \leqslant \dim_{\mathbb{C}} C_L(h) \ \forall h \in H$. Then $C_L(h_0) = C_L(H)$.*

Proof. Assume there is an $a \in H$ with $a \notin Z(C_L(h_0))$. We choose a basis of L by combining bases of its subspaces:

$C_L(a) \cap C_L(h_0)$	$e_1, \ldots, e_k$
$C_L(h_0)$	$e_1, \ldots, e_k, g_1, \ldots, g_r$
$C_L(a)$	$e_1, \ldots, e_k, f_1, \ldots, f_e$
$C_L(a) + C_L(h_0)$	$e_1, \ldots, e_k, g_1, \ldots, g_r, f_1, \ldots, f_e$
L	$e_1, \ldots, e_k, g_1, \ldots, g_r, f_1, \ldots, f_e, v_1, \ldots, v_s$

We will choose the v_i carefully however: since a and h_0 are semisimple, $\mathrm{ad}(a)$ and $\mathrm{ad}(h)$ are diagonalisable on L. Since $[a, h_0] = 0$ (since H is abelian), $\mathrm{ad}(a)$ and $\mathrm{ad}(h)$ are simultaneously diagonalisable. Hence we can find a basis of L consisting of eigenvectors of $\mathrm{ad}(a)$ and $\mathrm{ad}(h_0)$. Thus we choose v_i such that

$\mathrm{ad}(a)(v_i) = \alpha_i v_i$ and $\mathrm{ad}(h_0)(v_i) = \beta_i v_i$.
Take $\gamma \in \mathbb{C}$ with $\gamma \neq 0$ and $\gamma \neq -\frac{\alpha_i}{\beta_i} \forall i$. Look at $t = a + \gamma h_0 \in H$, let's see how $\mathrm{ad}(t)$ acts on the chosen basis of L:

$$\begin{aligned}
\mathrm{ad}(t)(e_i) &= [t, e_i] = [a + \gamma h_0, e_i] = [a, e_i] + \gamma[h_0, e_i] = 0 \\
\mathrm{ad}(t)(f_i) &= [t, f_i] = [a + \gamma h_0, f_i] = [a, f_i] + \gamma[h_0, f_i] \neq 0 \\
\mathrm{ad}(t)(g_i) &= [t, g_i] = [a + \gamma h_0, g_i] = [a, g_i] + \gamma[h_0, g_i] \neq 0 \\
\mathrm{ad}(t)(v_i) &= [t, v_i] = [a + \gamma h_0, v_i] = [a, v_i] + \gamma[h_0, v_i] \neq 0
\end{aligned}$$

$\dim_{\mathbb{C}} C_L(t) = k < k + r = \dim_{\mathbb{C}} C_L(h_0)$ as $a \notin Z(C_L(h_0))$. Therefore we have found $t \in H$ with $\dim_{\mathbb{C}} C_L(t) < \dim_{\mathbb{C}} C_L(h_0)$ which is a contradiction.
Therefore for all $a \in H$, $a \in Z(C_L(h_0))$, i.e. $H \subseteq Z(C_L(h_0))$. Hence $[H, C_L(h_0)] = \{0\}$ and thus $C_L(h_0) \subseteq C_L(H) \subseteq (h_0)$. ■

Using this lemma, we will show that $C_L(H) = H$.

Question. Why?

H is a Cartan subalgebra of L, so $\forall h \in H$, h is semisimple. H is abelian and acts on L by ad. For all $h \in H$ we have $\mathrm{ad}_L(H)\colon L \to L$, i.e. $\mathrm{ad}_L(H) \in \mathfrak{gl}(L)$. The $\mathrm{ad}_L(h)$ (for $h \in H$) are all simultaneously diagonalisable, so there is a basis of common eigenvectors of $\{\mathrm{ad}_L(h) \mid h \in H\}$. Let's take a common eigenvector $v \neq 0$. For all $h \in H$, $\mathrm{ad}(h)(v) = \alpha(h)v$. So we have a weight $\alpha\colon H \to \mathbb{C}$, $\alpha \in H^*$ and a weight space $L_\alpha \neq \{0\}$ corresponding to α. $L_\alpha = \{w \in L \mid [h, w] = \alpha(h)w \forall h \in H\}$. If $\alpha = 0$, $L_0 = \{w \in L \mid [h, w] = 0 \forall h \in H\} = C_L(H) \supseteq H$. Hence $L_0 \neq 0$.

H acts on L via ad. $L = L_\alpha \oplus L_\beta \oplus \cdots$, i.e. $L = \bigoplus_{\alpha \text{ a weight of } H} L_\alpha$. (By a weight of H we mean, of course, a weight of $\mathrm{ad}(H)$.) We define $\Phi = \{\alpha \mid \alpha$ is a weight of $H,\ \alpha \neq 0\}$. Now we can write $L = L_0 \oplus \bigoplus_{\alpha \in \Phi} L_\alpha$. Our goal is to show that this equals $H \oplus \bigoplus_{\alpha \in \Phi} L_\alpha$, called the Cartan decomposition of L with respect to H.

Claim VII.5. *For all $\alpha, \beta \in H^*$;*

1. $[L_\alpha, L_\beta] \subseteq L_{\alpha+\beta}$.
2. *If $\alpha + \beta \neq 0$, then $k(L_\alpha, L_\beta) = 0$.*
3. $L_0^\perp \neq 0$ *and $k|_{L_0}$ is non-degenerate.*

Proof. If α or β are not in Φ, then $L_\alpha = \{0\}$ or $L_\beta = \{0\}$. Then the claim obviously holds. So we may assume $L_\alpha \neq \{0\} \neq L_\beta$.

1. For all $x \in L_\alpha$, all $y \in L_\beta$ and all $h \in H$:

$$\begin{aligned}
\mathrm{ad}(h)([x, y]) &= [h, [x, y]] = -[x, [y, h]] - [y, [h, x]] = [x, [h, y]] + [[h, x], y] \\
&= [x, \beta(h)y] + [\alpha(h)x, y] = (\beta(h) + \alpha(h))[x, y] \\
&= (\alpha + \beta)(h)[x, y]
\end{aligned}$$

2. For all $x \in L_\alpha$, all $y \in L_\beta$:

$$\begin{aligned}\alpha(h)k(x,y) &= k(\alpha(h)x,y) = k([h,x],y) = -k([x,h],y) = -k(x,[h,y]) \\ &= -k(x,\beta(h)y) = -\beta(h)k(x,y)\end{aligned}$$

Hence $(\alpha(h)+\beta(h))k(x,y) = 0$. Since $\alpha+\beta \neq 0$, there exists $h \in H$ with $(\alpha+\beta)(h) \neq 0$. Therefore $k(x,y) = 0$.

3. Let $a \in L_0 \cap L_0^\perp$, then for all $x \in L_0$, $k(x,a) = 0$. For all $v \in L = L_0 \oplus \bigoplus_{\alpha\in\Phi} L_\alpha$, there exists $v_0 \in L_0$ and $v_\alpha \in L_\alpha$ $\forall \alpha \in \Phi$ such that $v = v_0 + \sum_{\alpha\in\Phi} v_\alpha$. But now $k(a,v) = \underbrace{k(a,v_0)}_{=0} + \sum_{\alpha\in\Phi} \underbrace{k(a,v_\alpha)}_{=0 \text{ by part 2.}} = 0$. Therefore $a \in L^\perp = \{0\}$ by Cartan's 2nd criterion. Hence $L_0 \cap L_0^\perp = \{0\}$. ■

Lemma VII.6. *$C_L(H) = H$.*

Proof. Let $h_0 \in H$ be as in VII.4 so that $C_L(h_0) = C_L(H)$. Recall $H \subseteq C_L(H)$ since H is abelian. For all $x \in C_L(H) = C_L(h_0)$, x has an abstract Jordan decomposition $x = d + n$ where $\mathrm{ad}(d)$ is diagonalisable, $\mathrm{ad}(n)$ is nilpotent and $[d,x] = 0 = [n,x]$. But $[x,h_0] = 0$ so by the remark at the end of the last section, $d, n \in C_L(h_0)$. But d is semisimple and $d \in C_L(h_0) = C_L(H)$ so $d \in H$ by maximality of H. $n \in C_L(h_0)$ and n is nilpotent (i.e. $\mathrm{ad}(n)$ is nilpotent). $x \in C_L(h_0)$ so $\mathrm{ad}(x)|_{C_L(h_0)}$ satisfies $\forall y \in C_L(h_0)$, $\mathrm{ad}(x)(y) = [x,y] \in C_L(h_0)$ and so $\mathrm{ad}(x)|_{C_L(h_0)} \colon C_L(h_0) \to C_L(h_0)$. Since $\forall y \in C_L(h_0) = C_L(H)$, $\mathrm{ad}(d)(y) = [d,y] = 0$, we have $\mathrm{ad}(x)|_{C_L(h_0)} = \mathrm{ad}(n)|_{C_L(h_0)}$.
We have shown that $\forall x \in C_L(h_0)$, $\mathrm{ad}(x)|_{C_L(h_0)} \colon C_L(h_0) \to C_L(h_0)$ is nilpotent. Therefore by Engel's Theorem, $C_L(h_0)$ is nilpotent, and hence solvable. Since L is semisimple, $L \cong \mathrm{ad}_L(L)$ and by restricting, $C_L(h_0) \cong \mathrm{ad}_L(C_L(h_0)) \subseteq \mathfrak{gl}(L)$. So $\mathrm{ad}_L(C_L(h_0))$ is a solvable subalgebra of $\mathfrak{gl}(L)$. Hence by Lie's Theorem, there is a basis of L such that the matrices representing $\mathrm{ad}_L(C_L(h_0))$ are all upper triangular. Since $\mathrm{ad}_L(n)$ is nilpotent, the matrix representing $\mathrm{ad}_L(n)$ is strictly upper triangular. Now for all $y \in C_L(h_0)$, $k|_{L_0}(n,y) = k(n,y) = \mathrm{tr}(\mathrm{ad}(n) \circ \mathrm{ad}(y)) = 0$. Hence $n \in L_0 \cap L_0^\perp = \{0\}$. Hence $n = 0$ and so $x = d \in H$. ■

We can now write $L = H \oplus \bigoplus_{\alpha\in\Phi} L_\alpha$, where $\Phi = \{\alpha \in H^* \mid \alpha \text{ is a weight of } H \text{ and } \alpha \neq 0\}$. Elements of Φ are called *roots* of L relative to H.

Remark. As $\dim_{\mathbb{C}} L < \infty$, $|\Phi| < \infty$, $\dim_{\mathbb{C}} L_\alpha < \infty$ and $\dim_{\mathbb{C}} H < \infty$.

Lemma VII.7. *For all $h \in H \setminus \{0\}$, there exists $\alpha \in \Phi$ such that $\alpha(h) \neq 0$.*

Proof. Assume $h_0 \in H \setminus \{0\}$ and that for all $\alpha \in \Phi$, $\alpha(h_0) = 0$. Now for $x \in L_\alpha$, $\mathrm{ad}(h_0)(x) = [h_0,x] = \alpha(h_0)x = 0$. Hence $[h_0, L_\alpha] = 0$ $\forall \alpha \in \Phi$, and so $[h_0, H] = 0$. Therefore $h_0 \in Z(L) = \{0\}$, and so $h_0 = 0$, which is a contradiction. ■

Corollary VII.8. $\langle \Phi \rangle_{\mathbb{C}} = H^*$.

Proof. Exercise. ■

Lemma VII.9. *If $\alpha \in \Phi$, then $-\alpha \in \Phi$.*

Proof. Assume $\alpha \in \Phi \not\ni -\alpha$. Now for all $\beta \in \Phi$, $\alpha + \beta \neq 0$ so $k(L_\alpha, L_\beta) = 0$ (by VII.5) and $\alpha + 0 \neq 0$ so $k(L_\alpha, L_0) = 0$ (also by VII.5). Hence $L_\alpha \subseteq L^\perp = \{0\}$, and so $L_\alpha = \{0\}$, which is a contradiction. ■

Lemma VII.10. *Let $\alpha \in \Phi$. There exists $t_\alpha \in H$ such that $\forall x \in L_\alpha$, $\forall y \in L_{-\alpha}$, $[x, y] = k(x, y)t_\alpha$. In fact, $\forall h \in H$, $k(t_\alpha, h) = \alpha(h)$.*

Proof. Fix $\alpha \in \Phi$, now if $x \in L_\alpha$, $y \in L_{-\alpha}$ then $[x, y] \in [L_\alpha, L_{-\alpha}] \subseteq L_{\alpha-\alpha} = H$. Look at $L_0 = H$ and consider $k|_{L_0}$. By VII.5, $k|_{L_0}$ is nondegenerate. We now have a nondegenerate symmetric bilinear $k|_H \colon H \times H \to \mathbb{C}$. By the "small review", there is an isomorphism $H \to H^*$ induced by $k|_H$. Take $\alpha \in \Phi \subseteq H^*$ and choose $t_\alpha \in H$ such that $k(t_\alpha, \cdot) = \alpha(\cdot)$. Now for all $h \in H$:

$$\begin{aligned} k(h, [x, y]) &= k([h, x], y) = k(\alpha(h)x, y) = \alpha(h)k(x, y) = k(t_\alpha, h)k(x, y) \\ &= k(k(x, y)t_\alpha, h) = k(h, k(x, y)t_\alpha) \end{aligned}$$

So $\forall h \in H$, $k(\underbrace{h}_{\in H}, \underbrace{[x, y]}_{\in H} - \underbrace{k(x, y)t_\alpha}_{\in H}) = 0$. So $[x, y] - k(x, y)t_\alpha \in H \cap H^\perp = \{0\}$. Hence $[x, y] = k(x, y)t_\alpha$. ■

Corollary VII.11. *For all $\alpha \in \Phi$, $[L_\alpha, L_{-\alpha}] = \langle t_\alpha \rangle_\mathbb{C}$.*

Proof. By VII.10, for all $x \in L_\alpha$, and all $y \in L_{-\alpha}$, $[x, y] = k(x, y)t_\alpha$. Hence $[L_\alpha, L_{-\alpha}] \subseteq \langle t_\alpha \rangle_\mathbb{C}$. But what if $[L_\alpha, L_{-\alpha}] = \{0\}$? If so, for all $x \in L_\alpha$, and all $y \in L_{-\alpha}$, $0 = [x, y] = k(x, y)t_\alpha$ and so $k(x, y) = 0$. But now $k(L_\alpha, L_{-\alpha}) = 0$, as well as $k(L_\alpha, L_\beta)$ for all $\beta \in \Phi \cup \{0\} \setminus \{-\alpha\}$. Hence $k(L_\alpha, L) = 0$ and so $L_\alpha \subseteq L^\perp = \{0\}$ which is a contradiction. Hence $[L_\alpha, L_{-\alpha}] = \langle t_\alpha \rangle_\mathbb{C}$. ■

Fix $\alpha \in \Phi$, now for all $x \in L_\alpha$ there is a $y \in L_{-\alpha}$ with $[x, y] = k(x, y)t_\alpha \neq 0$. Look at $M_\alpha = \langle x, y, [x, y] \rangle_\mathbb{C} = \langle x, y, t_\alpha \rangle$. M_α is a subalgebra of L (check!). $\dim_\mathbb{C} M_\alpha = 3$.

Assume first that $\alpha(t_\alpha) = 0$ (i.e. $\alpha([x, y]) = 0$). Then $t_\alpha, x] = \alpha(t_\alpha)x = 0$ and $t_\alpha, y] = -\alpha(t_\alpha)y = 0$. Therefore $t_\alpha \in Z(M_\alpha)$ and so $M_\alpha/Z(M_\alpha)$ is nilpotent and so is M_α (check!). $M_\alpha \cong \mathrm{ad}_L(M_\alpha) \subseteq \mathfrak{gl}(L)$, $\mathrm{ad}_L(M_\alpha)$ is nilpotent. Hence there is a basis of L such that $\mathrm{ad}_L(M_\alpha)$ is strictly upper triangular. Hence $\mathrm{ad}_L(t_\alpha)$ is strictly upper triangular, but $t_\alpha \in H$ so t_α is semisimple and $\mathrm{ad}_L(t_\alpha)$ is diagonalisable. Hence $t_\alpha = 0$ which is a contradiction, therefore $\alpha(t_\alpha) \neq 0$.

$\alpha(t_\alpha) = k(t_\alpha, t_\alpha) \neq 0$. Denote

$$e_\alpha := x \in L_\alpha, \quad h_\alpha := \frac{2}{k(t_\alpha, t_\alpha)} t_\alpha \in H, \quad f_\alpha := \frac{2}{k(t_\alpha, t_\alpha)k(x, y)} y \in L_{-\alpha}$$

$M_\alpha = \langle e_\alpha, h_\alpha, f_\alpha \rangle$.

$$\begin{aligned} [h_\alpha, e_\alpha] &= [\frac{2}{k(t_\alpha, t_\alpha)} t_\alpha, e_\alpha] = \frac{2}{k(t_\alpha, t_\alpha)}[t_\alpha, e_\alpha] = \frac{2}{k(t_\alpha, t_\alpha)}\alpha(t_\alpha)e_\alpha = 2e_\alpha \\ [h_\alpha, f_\alpha] &= \cdots = -2f_\alpha \\ [e_\alpha, f_\alpha] &= \cdots = h_\alpha \end{aligned}$$

Hence $M_\alpha \cong \mathfrak{sl}(2, \mathbb{C})$. This gives us:

Lemma VII.12. *For each $\alpha \in \Phi$ there exists a subalgebra $M_\alpha \subseteq L$ with $M_\alpha \cong \mathfrak{sl}(2, \mathbb{C})$.*

Lemma VII.13. $t_\alpha = -t_{-\alpha}$*,* $h_\alpha = -h_{-\alpha}$*,* $\alpha(h_\alpha) = 2$*.*

Proof. For all $h \in H$, $\alpha(h) = k(t_\alpha, h)$. Then $k(-t_{-\alpha}, h) = -k(t_{-\alpha}, h) = -(-\alpha(h)) = \alpha(h)$. So $k(t_\alpha + t_{-\alpha}, h) = 0$ for all $h \in H$. But VII.5 gives that $k|_H$ is nondegenerate and so $t_\alpha + t_{-\alpha} = 0$.
The rest is an exercise. ■

$\mathfrak{sl}(2, \mathbb{C}) = \langle e, f, h \rangle_{\mathbb{C}}$ is semisimple, so it has a Cartan subalgebra H. h is semisimple since $[\mathrm{ad}(h)]_{e,f,h} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}$. Since $\dim(\langle h \rangle \oplus L_\alpha \oplus L_{-\alpha}) = 3 = \dim(\langle h \rangle + L_\alpha + L_{-\alpha}) = \dim \mathfrak{sl}(2, \mathbb{C})$, $H = \langle h \rangle$.

VIII More Representation Theory

Let L be a semisimple Lie algebra over $\mathbb{C}$. Let V be a module of L. (V is a vector space, L acts on V, rest of definition V.2 holds.)

Definition VIII.1. A Lie module V of L is *irreducible* if V has no proper L-submodules.
i.e. $W \subseteq V$, W a submodule $\Rightarrow W = \{0\}$ or $W = V$.

Definition VIII.2. An L-module V is *completely reducible* if there are irreducible submodules $V_1, \dots, V_k$ such that $V = V_1 \oplus \cdots \oplus V_k$. Equivalently V is completely reducible if whenever W is an L-submodule of V there is an L-submodule W' such that $W \oplus W' = V$.

Weyl's Theorem. *If L is a semisimple Lie algebra over $\mathbb{C}$, then every L-module is completely reducible.*

No proof is given, since it would take far too long.

Consider $M = \mathfrak{sl}(2, \mathbb{C}) = \langle e, h, f \rangle_{\mathbb{C}}$. $\langle h \rangle = H$ is the Catran subalgebra of M. Let V be an irreducible M-module. Then there is a representation $\phi\colon M \to \mathfrak{gl}(V)$, $\phi(x)(v) = xv$. h is semisimple so it is its own abstract Jordan decomposition. Therefore by VI.18 $\phi(h)$ is its own Jordan decomposition, and therefore diagonalisable. Hence with respect to some basis of V, $[\phi(h)]$ is diagonal, i.e. there is a basis of V consisting of eigenvectors of h. Hence $V = \bigoplus_{\lambda \text{ is an eigenvalue of } h} V_\lambda$.

Lemma VIII.3. *If $\lambda \in \mathbb{C}$ is an eigenvalue of h and $v \in V_\lambda$, then:*

1. *either $ev = 0$ or $ev \in V_{\lambda+2}$*
2. *either $fv = 0$ or $fv \in V_{\lambda-2}$*

Proof.

1. $h(ev) = [h, e](v) + eh(v) = 2e(v) + e(hv) = 2ev + e(\lambda v) = 2ev + \lambda ev = (\lambda + 2)ev$.
2. Similarly. ∎

Since $\dim V < \infty$, the number of eigenvalues of h is finite, and we have finitely many summands in $V = \bigoplus_{\lambda \text{ is an eigenvalue of } h} V_\lambda$. On the other hand, if λ is an eigenvalue of h, we can look at $\lambda + 2, \lambda + 4, \dots$ in the light of the previous lemma. This tells us there must be an eigenvalue λ_0 of h such that $V_{\lambda_0} \neq \{0\} = V_{\lambda_0+2}$. We will call such a λ_0 a highest weight and we may choose $v_0 \in V_{\lambda_0} \setminus \{0\}$ and call v_0 a maximal eigenvector of λ_0.

Lemma VIII.4. *Let V be an irreducible M-module, λ_0 be a highest weight and v_0 be a maximal eigenvector. Set $v_1 := 0$ and for $i > 0$, $v_i := \frac{1}{i!} f^i v_0$. Then for $i \geqslant 0$:*

1. *$hv_i = (\lambda_0 - 2i)v_i$*
2. *$fv_i = (i + 1)v_{i+1}$*

3. $ev_i = (\lambda_0 - i + 1)v_{i-1}$

Proof. We give a sketch of why each is true, the details are uninspiring.

1. Follows from previous lemma.
2. Follows from definition of v_i.
3. Can be done by induction on i.

■

Let V be an irreducible M-module. Look at $\{v_0, v_1, \ldots\} \subseteq V$. For each i, v_i is an eigenvector of h with eigenvalue $\lambda_0 - 2i$. Hence for some $m \in \mathbb{N}$, $v_m \neq 0 = v_{m+1}$, and so $v_{m+j} = 0 \; \forall j \geqslant 1$.

Look at $\langle v_0, v_1, \cdots v_m \rangle_{\mathbb{C}} \subseteq V$. This subspace is h-invariant, e-invariant and f-invariant (by the previous lemma). This implies that $\langle v_0, v_1, \cdots v_m \rangle_{\mathbb{C}}$ is an M-submodule of V (check!), but V is irreducible, so $V = \langle v_0, v_1, \cdots v_m \rangle_{\mathbb{C}}$. Hence $\dim V = m + 1$. Let's write matrices of h, e, f with respect to this basis; the lemma clearly gives us:

$$[h]_{v_0,\ldots,v_m} = \begin{pmatrix} \lambda_0 & 0 & & & \\ 0 & \lambda_0 - 2 & 0 & & \\ & 0 & \lambda_0 - 4 & \ddots & \\ & & \ddots & \ddots & 0 \\ & & & 0 & \lambda_0 - 2m \end{pmatrix}$$

$$[e]_{v_0,\ldots,v_m} = \begin{pmatrix} 0 & \lambda_0 & & & \\ 0 & 0 & \lambda_0 - 1 & & \\ & 0 & 0 & \ddots & \\ & & \ddots & \ddots & \lambda_0 - m + 1 \\ & & & 0 & 0 \end{pmatrix}$$

$$[f]_{v_0,\ldots,v_m} = \begin{pmatrix} 0 & 0 & & & \\ 1 & 0 & 0 & & \\ & 2 & 0 & \ddots & \\ & & \ddots & \ddots & 0 \\ & & & m & 0 \end{pmatrix}$$

But note also that $(\lambda_0 - m)v_m = ev_{m+1} = e0 = 0$. Hence (since $v_m \neq 0$) $\lambda_0 = m$. We have now proved the following theorem:

Theorem VIII.5. *If V is an irreducible module for $M = \mathfrak{sl}(2, \mathbb{C})$, then:*

1. *Relative to h, $V = \bigoplus_{\mu = m, m-2, \ldots, -m} V_\mu$ where $\dim V = m + 1$, $\dim V_\mu = 1$.*
2. *The action of M on V is described explicitly (for some basis) by the ma-*

trices

$$[h] = \begin{pmatrix} m & 0 & & & \\ 0 & m-2 & 0 & & \\ & 0 & \ddots & \ddots & \\ & & \ddots & 2-m & 0 \\ & & & 0 & -m \end{pmatrix}$$

$$[e] = \begin{pmatrix} 0 & m & & & \\ 0 & 0 & m-1 & & \\ & 0 & \ddots & \ddots & \\ & & \ddots & 0 & 1 \\ & & & 0 & 0 \end{pmatrix}$$

$$[f] = \begin{pmatrix} 0 & 0 & & & \\ 1 & 0 & 0 & & \\ & 2 & \ddots & \ddots & \\ & & \ddots & 0 & 0 \\ & & & m & 0 \end{pmatrix}$$

and hence for all $m \in \mathbb{N}$ *there is (up to an equivalence of modules) at most* 1 *irreducible* M*-module of dimension* $m+1$.

IX More on Semisimple Lie Algebras

Let L be a semisimple Lie algebra over $\mathbb{C}$, then we know $L = H \oplus \bigoplus_{\alpha\in\Phi} L_\alpha$.

Proposition IX.1. *Let $\alpha \in \Phi$, if $c\alpha \in \Phi$ for some $c \in \mathbb{C}$, then $c = \pm 1$.*

Proof. Let $W := \langle H, L_{c\alpha} \mid c\alpha \in \Phi\rangle_{\mathbb{C}} = \langle L_{c\alpha} \mid c \in \mathbb{C}\rangle_{\mathbb{C}} \subseteq L$. There exists $M_\alpha \subseteq L$ such that $M_\alpha = \langle e_\alpha, h_\alpha, f_\alpha\rangle_{\mathbb{C}} \cong \mathfrak{sl}(2,\mathbb{C})$. For any $x \in M_\alpha$, $x = ae_\alpha + bf_\alpha + dh_\alpha$. Now for all $h \in H$, $[x,h] = a[e_\alpha,h] + b[f_\alpha,h] + d[h_\alpha,h] = -a\alpha(h)e_\alpha + b(-\alpha(h))f_\alpha + 0 \in W$. Also for $y \in L_{c\alpha}$ where $c\alpha \in \Phi$, $[x,y] = a\underbrace{[e_\alpha,y]}_{\in L_{c\alpha+\alpha}} + b\underbrace{[f_\alpha,y]}_{\in L_{c\alpha-\alpha}} + d\underbrace{[h_\alpha,y]}_{=\alpha(h_\alpha)y} \in W$. We now have that for all $x \in M_\alpha$ and all $w \in W$, $[x,w] \in W$. W is a vector space over $\mathbb{C}$ and so defining $xw := [x,w] \in W$ makes W into an M_α-module. Let's study this module.
Since $\Phi \ni \alpha\colon H \to \mathbb{C}$, $H_\alpha := \ker\alpha \subseteq H$ has dimension $\dim H_\alpha = \dim H - 1$. As $\alpha(h_\alpha) = 2 \neq 0$, $H = H_\alpha \oplus \langle h_\alpha\rangle$. For all $x \in M_\alpha$, and all $h \in H_\alpha \subseteq W$, $xh = [x,h] = -a\alpha(h)e_x - b\alpha(h)f_\alpha = 0$. This means that H_α is a trivial M_α-module which is a submodule of W. Consider $M_\alpha = \langle e_\alpha, h_\alpha, f_\alpha\rangle_{\mathbb{C}} \subseteq W$. Since for all $x, w \in M_\alpha$, $xw = [x,w] \in M_\alpha$, M_α is an M_α-module. In fact, M_α is an irreducible M_α-module (check!). If $h \in H_\alpha \cap M_\alpha$ then $\alpha(h) = 0$ so $h \in H$, but h would be in the Cartan subalgbra of M_α which is $\langle h_\alpha\rangle$ and so $h = 0$. Hence $H_\alpha \cap M_\alpha = \{0\}$ and so $W \supseteq H_\alpha \oplus M_\alpha$, which is an M_α-submodule of W. Since W is an M_α-module, it is completely reducible by Weyl's Theorem. So, by the definition, there is an M_α-submodule W_α of W such that $W = H_\alpha \oplus M_\alpha \oplus M_\alpha \subseteq L$. Our goal now is to show that $W_\alpha = \{0\}$.
W_α is an M_α-module so it contains an irreducible submodule W_m of dimension $m+1$. By theorem VIII.5, there is a basis of W_m such that $[h_\alpha] = \operatorname{diag}(m, m-2, \ldots, -m)$.
Suppose $m = 2l$. Then $[h_\alpha] = \operatorname{diag}(2l, 2l-2, \ldots, 0, \ldots, -2l)$ and so there exists $0 \neq w \in W_m \subseteq W$ such that $h_\alpha w = [h_\alpha, w] = 0w = 0$. Hence $w \in L_{c\alpha} \Rightarrow [h_\alpha, w] = (c\alpha)(h_\alpha)w = c\alpha(h_\alpha)w = 2cw \Rightarrow c = 0$. So $w \in L_0 = H \subseteq H_\alpha \oplus M_\alpha$, which means $w \in (H_\alpha \oplus M_\alpha) \cap W_\alpha$ which is a contradiction. Hence m is odd.
We claim that for all $\alpha \in \Phi$, $2\alpha \notin \Phi$. To prove this, assume that $\alpha, 2\alpha \in \Phi$, then $2\alpha(h_\alpha) = 4$ is an eigenvalue of h_α (in its action on W). But what are the eigenvalues of h_α? We consider $W = H_\alpha \oplus M_\alpha \oplus W_\alpha$. For all $h \in H$, $[h_\alpha, h] = 0h$ so the eigenvalues on H_α are 0. $[h_\alpha, h_\alpha] = 0$, $[h_\alpha, e_\alpha] = 2e_\alpha$ and $[h_\alpha, f_\alpha] = -2f_\alpha$ so the eigenvalues on M_α are $0, \pm 2$. The eigenvalues on W_α are all odd. Therefore 4 cannot be an eigenvalue of h_α, this contradiction proves the claim.
We now know that (in its action on W_m $(m = 2l+1)$) $[h_\alpha] = \operatorname{diag}(2l+1, 2l-1, \ldots, 1, \ldots, -2l-1)$, so there exists $0 \neq w \in W_m$ such that $h_\alpha w = [h_\alpha, w] = w$. We now have $w \in L_{c\alpha} \Rightarrow [h_\alpha, w] = (c\alpha)(h_\alpha)w = c\alpha(h_\alpha)w = 2cw \Rightarrow c = \frac{1}{2}$. Hence $\frac{1}{2}\alpha, \alpha \in \Phi$, contradicting the claim.
Hence $W_\alpha = \{0\}$ and so $\langle H, L_{c\alpha} \mid c\alpha \in \Phi\rangle_{\mathbb{C}} = W = H_\alpha \oplus M_\alpha = H_\alpha \oplus \langle h_\alpha\rangle \oplus L_\alpha \oplus L_{-\alpha} = H \oplus L_\alpha \oplus L_{-\alpha}$, completing the proof. ■

Corollary IX.2. *For all $\alpha \in \Phi$, $\dim L_\alpha = 1$.*

Proof. $H_\alpha \oplus M_\alpha = H \oplus L_\alpha \oplus L_{-\alpha}$ ■

So $L = H \oplus \bigoplus_{\alpha\in\Phi} L_\alpha$ gives us $\dim L = \dim H + |\Phi|$ and $|\Phi|$ is even.

Lemma IX.3. *For all $\alpha, \beta \in \Phi$, $\beta(h_\alpha) \in \mathbb{Z}$.*

Proof. $\langle h_\alpha \rangle$ acts on L via $\mathrm{ad}_L(h_\alpha)$. In fact, M_α acts on L via ad_L, so L is an M_α-module. By Weyl's Theorem L decomposes as a sum of irreducible M_α-submodules. On each of these submodules, the eigenvalues of h_α are $m, m-1, \ldots, -m$. So all of the eigenvalues of h_α are integers. Hence $\beta(h_\alpha)$, which is an eigenvalue of h_α, is an integer. ■

Lemma IX.4. *Let $\alpha, \beta \in \Phi$, $\beta \neq \pm\alpha$. Then*

1. *There exist $q, r \in \mathbb{N}$ (including 0) where $r - q = \beta(h_\alpha)$, such that for all $i \in \mathbb{Z}$, $\beta + i\alpha \in \Phi \iff -r \leqslant i \leqslant q$.*
2. *$\beta - \beta(h_\alpha)\alpha \in \Phi$.*

Proof. $W := \langle L_{\beta+i\alpha} \mid i \in \mathbb{Z},\ \beta+i\alpha \in \Phi\rangle_{\mathbb{C}} \subseteq L$. Since α is a root, we have a subalgebra of L, $M_\alpha = \langle e_\alpha, h_\alpha, f_\alpha \rangle \cong \mathfrak{sl}(2, \mathbb{C})$. We find $[L_\alpha, L_{\beta+i\alpha}] \subseteq L_{\beta+(i+1)\alpha} \subseteq W$, $[L_{-\alpha}, L_{\beta+i\alpha}] \subseteq L_{\beta+(i-1)\alpha} \subseteq W$ and $[H, L_{\beta+i\alpha}] \subseteq L_{\beta+i\alpha} \subseteq W$ and so $[M_\alpha, W] \subseteq W$. So W is an M_α-module via $mw := [m, w]$. What are the eigenvalues of h_α? For $x \in L_{\beta+ia}$, $h_\alpha x = [h_\alpha, x] = (\beta + i\alpha)(h_\alpha)x = \beta(h_\alpha) + 2i \in \mathbb{Z}$. All the eigenvalues are therefore integers, and all are equivalant modulo 2. Furthermore each value occurs (at most) for a single 1-dimensional subspace. Hence W cannot be the sum of 2 or more irreducible submodules, since it would then contain 0 or 1 as an eigenvalue twice. Hence W is irreducible. Thus the eigenvalues of h_α are precisely $m, m-2, \ldots, -m$. So we can pick $q, r \in \mathbb{N}$ with $m = \beta(h_\alpha) + 2q$, $-m = \beta(h_\alpha) - 2r$; this is so that $\beta + q\alpha \in \Phi \not\ni \beta + (q+1)\alpha$ and $\beta - r\alpha \in \Phi \not\ni \beta - (r+1)\alpha$.
We now consider add the equations $m = \beta(h_\alpha) + 2q$ and $-m = \beta(h_\alpha) - 2r$, giving $0 = 2\beta(h_\alpha) + 2q - 2r$, and so $\beta(h_\alpha) = r - q$. We now have $\beta - \beta(h_\alpha)\alpha = \beta - (r-q)\alpha = \beta + (q-r)\alpha$, and $-r \leqslant q - r \leqslant q$, completing the proof. ■

Exercise. Let $\alpha, \beta \in \Phi$, if $\alpha + \beta \in \Phi$, then $[L_\alpha, L_\beta] = L_{\alpha+\beta}$.

Lemma IX.5. *If $\alpha, \beta \in \Phi$, then $k(h_\alpha, h_\beta) \in \mathbb{Z}$ and $k(t_\alpha, t_\beta) \in \mathbb{Q}$.*

Proof.

$$k(h_\alpha, h_\beta) = \mathrm{tr}(\mathrm{ad}(h_\alpha) \circ \mathrm{ad}(h_\beta)) = \sum_{\gamma \in \Phi} \gamma(h_\alpha)\gamma(h_\beta) \in \mathbb{Z}$$

$$k(t_\alpha, t_\beta) = k\Big(\frac{k(t_\alpha, t_\alpha)}{2} h_\alpha, \frac{k(t_\beta, t_\beta)}{2} h_\beta\Big) = \frac{1}{4} k(t_\alpha, t_\alpha) k(t_\beta, t_\beta) k(h_\alpha, h_\beta)$$

Now, using this with $\beta = \alpha$ gives $k(t_\alpha, t_\alpha) = k(t_\alpha, t_\alpha)^2 \frac{1}{4} k(h_\alpha, h_\alpha)$ and hence $k(t_\alpha, t_\alpha) \in \mathbb{Q}$. Similarly $k(t_\beta, t_\beta) \in \mathbb{Q}$. Hence $k(t_\alpha, t_\beta) \in \mathbb{Q}$. ■

We have that $k|_H$ is a non-degenerate symmetric bilinear form. Hence the map $H \to H^*$ defined by $h \mapsto k(h, \cdot)$ is an isomorphism. Now we have

$$\forall \phi \in H^*\ \exists t_\phi \in H \text{ such that } t_\phi \mapsto k(t_\phi, \cdot) = \phi(\cdot)$$

Look at H^*; for all $\phi, \theta \in H^*$ let $(\phi, \theta) := k(t_\phi, t_\theta)$. k is a non-degenerate symmetric bilinear form, and therefore $(\cdot, \cdot)\colon H^* \to H^*$ is also a non-degenerate symmetric bilinear form (check!). For all $\alpha, \beta \in \Phi$, $(\alpha, \beta) = k(t_\alpha, t_\beta) \in \mathbb{Q}$.

Since we have $\langle\Phi\rangle = H^*$, there exist roots $\alpha_1, \ldots, \alpha_l \in \Phi$ which form a basis of H^*. For all $\beta \in \Phi$, $\beta = c_1\alpha_1 + \cdots + c_l\alpha_l$ for some $c_i \in \mathbb{C}$. Look at $(\beta, \alpha_j) = \sum_{i=1}^{l} c_i(\alpha_i, \alpha_j) = \sum_{i=1}^{l} c_i(\alpha_j, \alpha_i)$.

$$\mathbb{Q}^{l,1} \ni \begin{pmatrix} (\beta,\alpha_1) \\ \vdots \\ (\beta,\alpha_l) \end{pmatrix} = \begin{pmatrix} (\alpha_1,\alpha_1) & \cdots & (\alpha_1,\alpha_l) \\ \vdots & \ddots & \vdots \\ (\alpha_l,\alpha_1) & \cdots & (\alpha_l,\alpha_l) \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_l \end{pmatrix}$$

$t_{\alpha_1}, \ldots, t_{\alpha_l}$ is a basis of H. Hence $(k(t_{\alpha_i}, t_{\alpha_j}))_{i,j}$ is the matrix of a nondegenerate symmetric bilinear form, and so is invertible in $\mathbb{Q}^{l,l}$. Since $((\alpha_i, \alpha_j))_{i,j} = (k(t_{\alpha_i}, t_{\alpha_j}))_{i,j}$, we find that $\begin{pmatrix} c_1 \\ \vdots \\ c_l \end{pmatrix} \in \mathbb{Q}^{l,1}$.

We can now define $E := \mathbb{R}[\alpha_1, \ldots, \alpha_l] \supseteq \Phi$. E is a real vector space, so we may consider the restriction $(\cdot,\cdot)|_{E\times E}$. For all $\theta \in E$:

$$\begin{aligned}(\theta,\theta) &= k(t_\theta, t_\theta) = \operatorname{tr}(\operatorname{ad}(t_\theta) \circ \operatorname{ad}(t_\theta)) = \sum_{\gamma\in\Phi} \gamma(t_\theta)\gamma(t_\theta) = \sum_{\gamma\in\Phi} \gamma(t_\theta)^2 \\ &= \sum_{\gamma\in\Phi} k(t_\gamma, t_\theta)^2 = \sum_{\gamma\in\Phi} \underbrace{(\gamma,\theta)}_{\in\mathbb{R}}{}^2 \geqslant 0\end{aligned}$$

and

$$(\theta,\theta) = 0 \iff \sum_{\gamma\in\Phi} \gamma(t_\theta)^2 = 0 \iff \gamma(t_\theta) = 0 \ \forall\gamma \in \Phi \iff t_\theta = 0 \iff \theta = 0$$

so $(\cdot,\cdot)|_{E\times E}$ is an inner product.

We have shown that:

- E is a vector space over $\mathbb{R}$ with a real inner product.
- $E = \langle\Phi\rangle_{\mathbb{R}}$, $0 \notin \Phi$.
- $\forall\alpha \in \Phi$, $c\alpha \in \Phi \iff c = \pm 1$.
- $\forall\alpha, \beta \in \Phi$, $2\frac{(\beta,\alpha)}{(\alpha,\alpha)} = k\left(t_\beta, \frac{2}{k(t_\alpha,t_\alpha)}t_\alpha\right) = k(t_\beta, h_\alpha) = \beta(h_\alpha) \in \mathbb{Z}$, and $\beta - 2\frac{(\beta,\alpha)}{(\alpha,\alpha)}\alpha = \beta - \beta(h_\alpha) \in \Phi$.

X Root Systems

Let E be a vector space over $\mathbb{R}$, $\dim_{\mathbb{R}} E < \infty$ with a real-valued inner product $(\cdot,\cdot)\colon E \times E \to \mathbb{R}$. For all $v \in E \setminus \{0\}$ define $\sigma_v\colon E \to E$ by $\sigma_v\colon x \mapsto x - 2\frac{(x,v)}{(v,v)}v$ which is a linear map. Clearly $\sigma_v(v) = -v$ and if $y \perp v$ then $\sigma_v(y) = y - 0v = y$, so σ_v is a reflection in $\langle v \rangle^{\perp}$, which is invertible (in fact, is an involution).

Notation. $\langle x, v \rangle := 2\frac{(x,v)}{(v,v)}$.

Remark. For all $x, y, v \in E$, $(\sigma_v(x), \sigma_v(y)) = (x, y)$.

Definition X.1. Let $R \subseteq E$. Then R is a *root system* if:

(R1) $|R| < \infty$, $0 \notin R$, $\langle R \rangle_{\mathbb{R}} = E$.

(R2) If $\alpha \in R$ then $r\alpha \in R \iff r = \pm 1$.

(R3) $\forall \alpha \in R$, $\sigma_\alpha(R) \subseteq R$.

(R4) $\forall \alpha, \beta \in R$, $\langle \alpha, \beta \rangle \in \mathbb{Z}$.

Example. Let L be a semisimple Lie algebra over $\mathbb{C}$, H be a Cartan subalgebra, and Φ be the set of roots. Then Φ is a root system by the properties at the end of the last chapter.

From now on, assume R is a root system of E.

Lemma X.2. *For all $\alpha, \beta \in R$, $\beta \neq \pm\alpha$, then $\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle \in \{0, 1, 2, 3\}$.*

Proof. For all $v, w \in E$, $(v,w)^2 = (v,v)(w,w)\cos^2\theta$ where $\theta = \widehat{v,w}$ is the angle between v and w. This gives $\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle = 2\frac{(\alpha,\beta)}{(\beta,\beta)} 2\frac{(\beta,\alpha)}{(\alpha,\alpha)} = 4\cos^2(\widehat{\alpha,\beta}) \in [0,4] \cap \mathbb{Z} = \{0,1,2,3,4\}$.
Now if $\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle = 4$ then $\cos^2(\widehat{\alpha,\beta}) = 1$ and so $\beta = \pm\alpha$ which is a contradiction. Hence $\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle \in \{0,1,2,3\}$. ■

Take $\alpha, \beta \in R$ and assume without loss of generality that $(\beta,\beta) \geqslant (\alpha,\alpha)$. This gives $|\langle \alpha, \beta \rangle| = 2\frac{|(\alpha,\beta)|}{(\beta,\beta)} \leqslant 2\frac{|(\beta,\alpha)|}{(\alpha,\alpha)} = |\langle \beta, \alpha \rangle|$. We can now construct a table of the situation for each value of $\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle$:

$\langle \alpha,\beta \rangle \langle \beta,\alpha \rangle$	$\langle \alpha,\beta \rangle$	$\langle \beta,\alpha \rangle$	$\widehat{\alpha,\beta}$	$\frac{\|\beta\|}{\|\alpha\|}$
0	0	0	$\pi/2$	–
1	1	1	$\pi/3$	1
1	-1	-1	$2\pi/3$	1
2	1	2	$\pi/4$	$\sqrt{2}$
2	-1	-2	$3\pi/4$	$\sqrt{2}$
3	1	3	$\pi/6$	$\sqrt{3}$
3	-1	-3	$5\pi/6$	$\sqrt{3}$

Lemma X.5. *If $\alpha, \beta \in R$ and $(\beta,\beta) \geqslant (\alpha,\alpha)$ then:*

1. *If $\widehat{\alpha,\beta} > \pi/2$ then $\alpha + \beta \in R$.*
2. *If $\widehat{\alpha,\beta} < \pi/2$ then $\alpha - \beta \in R$.*

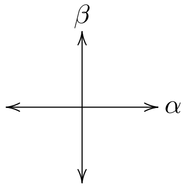
Proof. $\sigma_\beta(\alpha) = \alpha - \langle \alpha, \beta \rangle \beta \in R$ by (R3). If $\widehat{\alpha, \beta} > \pi/2$ then $\langle \alpha, \beta \rangle = -1$ and so $\sigma_\beta(\alpha) = \alpha - (-\beta) = \alpha + \beta \in R$. If $\widehat{\alpha, \beta} < \pi/2$ then $\langle \alpha, \beta \rangle = 1$ and so $\sigma_\beta(\alpha) = \alpha - \beta \in R$. ■

Examples.

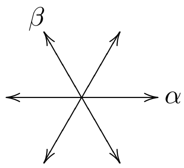
1. If $\dim E = 1$ then $R = \{\pm\alpha\}$, this is called Type A_1.

2. If $\dim E = 2$, take $\alpha, \beta \in R$, and suppose R is the smallest root system containing α and β:

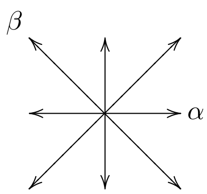
   (a) If $\theta = \pi/2$ then we know nothing about $\|\beta\|$ and $\|\alpha\|$, but $\sigma_\alpha(\beta) = \beta$, $\sigma_\beta(\alpha) = \alpha$, so $R = \{\pm\alpha, \pm\beta\}$ is a Type $A_1 \times A_1$ root system.



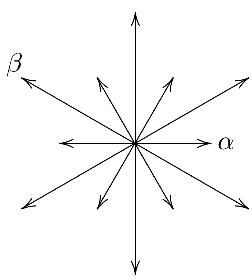
   (b) If $\theta = \frac{2\pi}{3}$, we have $\|\beta\| = \|\alpha\|$. By X.5, $\alpha + \beta \in R$. Now consideration of (R3) will not generate any new roots. So we can have $R = \{\pm\alpha, \pm\beta, \pm(\alpha + \beta)\}$. This is type A_2.



   (c) If $\theta = \frac{3\pi}{4}$, we have $\frac{\|\beta\|}{\|\alpha\|} = \sqrt{2}$. By X.5, $\alpha + \beta \in R$. Also $R \ni \sigma_\alpha(\beta) = \beta - \langle \beta, \alpha \rangle = \beta + 2\alpha$. Again we get no additional roots, so $R = \{\pm\alpha, \pm\beta, \pm(\beta + \alpha), \pm(\beta + 2\alpha)\}$. This is type B_2.



   (d) The remaining case, $\theta = \frac{5\pi}{6}$ and $\frac{\|\beta\|}{\|\alpha\|} = \sqrt{3}$ is an exercise, and gives a G_2 root system.



Definition X.6. R is *irreducible* if you cannot write $R = R_1 \cup R_2$ where each R_i is a root system in $\langle R_i \rangle$, $R_1 \cap R_2 = \emptyset$ and $(\alpha_1, \alpha_2) = 0$ whenever $\alpha_i \in R_i$.

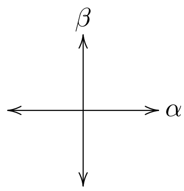
Lemma X.7. *If R is a root system, then either R is irreducible or $R = R_1 \sqcup R_2 \sqcup \cdots \sqcup R_l$ where each R_i is a root system of $E_i = \langle R_i \rangle$, $E = E_1 \oplus E_2 \oplus \cdots \oplus E_l$ and $E_i \perp E_j$ for $i \neq j$.*

Sketch proof: One may introduce an equivalence relation $\sim$ on R by $\alpha \sim \beta$ iff $\exists \gamma_1, \ldots \gamma_t$ such that ¡rubbed out!.

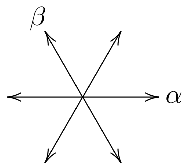
Definition X.8. Let R be a root system in E, and $B \subseteq R$. Then B is a *base* for R if:

1. B is a basis for E
2. $\forall \alpha \in R$, $\alpha = \sum_{\beta \in B} a_\beta \beta$ where $a_\beta \in \mathbb{Z}$ and either $a_\beta \geqslant 0$ $\forall \beta \in B$ (in which case α is called a *positive root*) or $a_\beta \leqslant 0$ $\forall \beta \in B$ (in which case α is called a *negative root*).

Examples.

2. (a) 

$B = \{\alpha, \beta\}$.

   (b) 

$B = \{\alpha, \beta\}$.

Theorem X.9. *Every root system has a base.*

Sketch proof. It is easy to satisfy 1. since $\langle R \rangle = E$. Algorithm for 2.: choose $v \in E$ such that $(v, \alpha) \neq 0$ $\forall \alpha \in R$. Then let the positive roots be $R^+ = \{\alpha \in R \mid (v, \alpha) > 0\}$. Define B as a minimal subset of R^+ satisfying $\forall \alpha \in R^+$ $\exists \beta_1, \ldots, \beta_k \in B$ s.t. $\alpha = \beta_1 + \cdots + \beta_k$. ■

Weyl Group of R

For all $\alpha \in R$, we have that $\sigma_\alpha \colon E \to E$ is the reflection in the hyperplane $\langle \alpha \rangle^\perp$. Hence (with respect to some basis of E) $[\sigma_\alpha] = \begin{pmatrix} -1 & & 0 \\ & 1 & \\ 0 & & \ddots \\ & & & 1 \end{pmatrix}$. Therefore $\sigma_\alpha \in GL(E)$. We can now define $W = W(R) := \langle \sigma_\alpha \mid \alpha \in R \rangle \subseteq GL(E)$. $W(R)$ is the *Weyl group* of R, and is a subgroup of $GL(E)$.

For all $\alpha, \beta \in R$, $\sigma_\alpha(\beta) \in R$ by (R3). Hence for all $\alpha \in R$, $\sigma_\alpha(R) = R$, and so for all $g \in W(R)$, $g(R) = R$. Therefore there is a natural homomorphism $f \colon W(R) \to \Sigma_{|R|}$, the group of permutations of $|R|$ letters.

If $g \in \ker f$ then for all $\beta \in R$, $g(\beta) = \beta$ but $\langle R \rangle = E$ so $g = \mathrm{id}_E$. Hence $\ker f = \{\mathrm{id}\}$. Therefore $W(R) \cong f(W(R)) \leqslant \Sigma_{|R|}$; in particular, $W(R)$ is finite.

Facts about $W(R)$:

1. If B is a base of R, then $W(R) = \langle \sigma_\alpha \mid \alpha \in B \rangle$.
2. For all $\alpha \in R$, There exist $g \in W(R)$ and $\alpha_i \in B$ such that $\alpha = g(\alpha_i)$.

Theorem X.10. *Let R be a root system and B, B' be bases of R. Then there exists $g \in W(R)$ such that $B' = \{g(\alpha) \mid \alpha \in B'\}$.*

(No proof)

Let R be a root system, and $B = \{\alpha_1, \ldots, \alpha_l\}$ be a base. The *Cartan matrix* of R is the matrix $(\langle \alpha_i, \alpha_j \rangle)_{l \times l}$. Since for all $\beta \in R$, $\langle \sigma_\beta(\alpha_i), \sigma_\beta(\alpha_j) \rangle = 2\frac{(\sigma_\beta(\alpha_i), \sigma_\beta(\alpha_j))}{(\sigma_\beta(\alpha_i), \sigma_\beta(\alpha_i))} = 2\frac{(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} = \langle \alpha_i, \alpha_j \rangle$ we have that for all $g \in W(R)$, $\beta \in R$, $\langle g(\alpha_i), g(\alpha_j) \rangle = \langle \alpha_i, \alpha_j \rangle$. This, together with X.10 tells us that the Cartan matrix is independant of the choice of base (but not the order).

Examples.

2.(a) The Cartan matrix of $A_1 \times A_1$ is $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$.

2.(b) The Cartan matrix of A_2 is $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$.

2.(c,d) Exercise: compute the Cartan matrices of B_2 and G_2.

Given R we define $\Delta(R)$ to be the (multi)graph whose vertices are elements of B, and the number of edges between α and β is $d_{\alpha\beta} = \langle \alpha, \beta \rangle \langle \beta, \alpha \rangle$. If $\|\alpha\| \neq \|\beta\|$ then the edge is directed towards the shorter root (denoted by an arrowhead in the middle of the edge(s)). This graph is called the *Dynkin diagram* of R. Notice that the Dynkin diagram contains as much information as the Cartan matrix, since $\langle \alpha_i, \alpha_j \rangle$ can be determined from the line(s) connecting the corresponding vertices in the diagram.

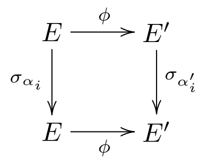
Examples.

1. $\Delta(A_1) =$ ○.
2. (a) $\Delta(A_1 \times A_1) =$ ○ ○.
   (b) $\Delta(A_2) =$ ○—○.
   (c) $\Delta(B_2) =$ ○⇒○.
   (d) $\Delta(G_2) =$ ○⇛○.

Theorem X.11. $R \cong R' \iff \Delta(R) \cong \Delta(R')$.

Proof. Clearly $R \cong R' \Rightarrow \Delta(R) \cong \Delta(R')$.
We want to show that we can construct R from $\Delta(R)$. Take a vector space E with the vertices of $\Delta(R)$ $(\alpha_1, \ldots, \alpha_k)$ as a basis. Construct an inner product on E from the Cartan matrix. Let W be the group generated by the simple reflections σ_{α_i}. Take R to be the image under W of the simple roots $\alpha_1, \ldots, \alpha_k$. Assume $\Delta(R) = \Delta(R')$. Define $\phi\colon E \to E'$ by $\phi(v_i) = v_i'$, where v_i is a vertex of $\Delta(R)$ (and hence an element of the basis of E) and v_i' is the corresponding vertex of $\Delta(R')$, and extend by linearity. This preserves the inner product,

$\langle \alpha, \beta \rangle = \langle \phi(\alpha), \phi(\beta) \rangle$, since this holds for the basis. This implies that the following diagram commutes:



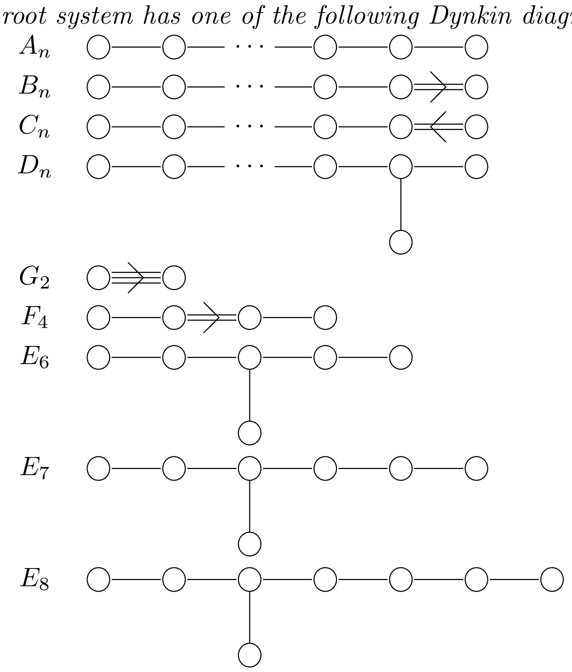
and hence $\phi(g\alpha) = g\phi(\alpha)$ for all g in the Weyl group.

Fact. *If $\beta \in R$, then $\beta \in g\alpha_i$ for some $g \in W$ and some simple root α_i.*

Thus $\phi(R) \subseteq R'$, and similarly $\phi^{-1}(R') \subseteq R$. Hence ϕ is an isomorphism of root systems. ■

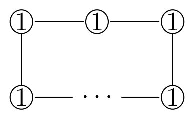
R is irreducible iff $\Delta(R)$ is connected. This is because if $\Delta(R) = \Delta(R_1) \sqcup \Delta(R_2)$ then $E = \langle R_1 \rangle \oplus \langle R_2 \rangle$ with $(v_1, v_2) = 0$ for $v_i \in E_i := \langle R_i \rangle$ and so R reduces as $R = R_1 \cup R_2$. Conversely if $R = R_1 \cup R_2$ with $(\alpha_1, \alpha_2) = 0$ for $\alpha_i \in R_i$ then there are no lines in $\Delta(R)$ between the simple roots of R_1 and those of R_2.

Theorem X.12 (Classification of irreducible root systems)**.** *Every irreducible root system has one of the following Dynkin diagrams:*



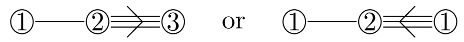
Proof. We know that $(\cdot, \cdot)$ is positive definite. We can show that for any other diagram there would be a vector $v \neq 0$ with $(v, v) \leqslant 0$. We will give the proof by presenting, at each step, a diagram which we intend to show cannot occur as a subgraph of $\Delta(R)$. We label each vertex v_i of the graph with a real number λ_i, and consider $E \ni v = \lambda_1 v_1 + \cdots + \lambda_k v_k$ since the v_i are elements of the basis of E. It will always be apparent that $(v_i, v) = \sum_{j=1}^{k} \lambda_j (v_i, v_j) \leqslant 0$ and hence $(v, v) = \sum_{i=1}^{k} \lambda_i (v_i, v) \leqslant 0$.

If our diagram contains a loop (possibly including multiple edges), we can construct v by



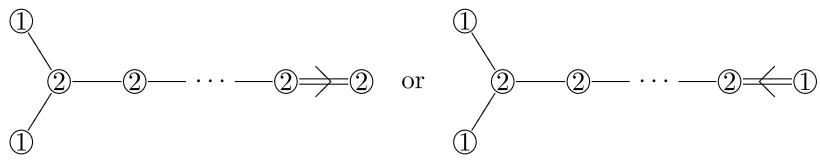
which tells us that the diagram must be a tree. (For this one only, we use that replacing single edges with multiple ones will only decrease the value of (v, v) and so we need not worry about whether the lines in the loop are single or multiple.)

If we have a triple edge, we cannot have any more edges, using



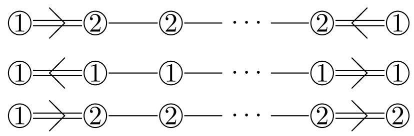
so if we have a triple edge, then the diagram must be G_2.

If we have both a double edge and a branch we use

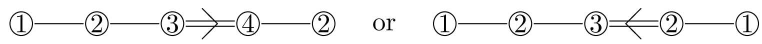


so if we have a double edge, we must have a straight line.

Consideration of these three diagrams

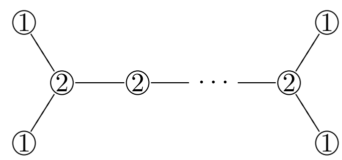


tells us that we cannot have two double edges, and

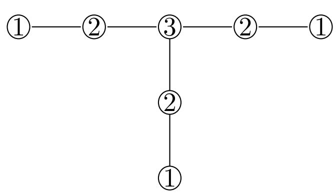


indicates that a double edge can only be at the end (i.e. in B_n or C_n) or in F_4.

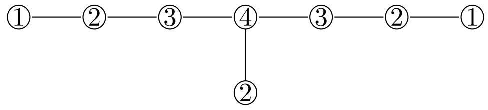
We have now covered all possiblities involving multiple edges, so suppose from now on that all edges are single. Two triple points (or a quadruple or higher point) are not allowed because of



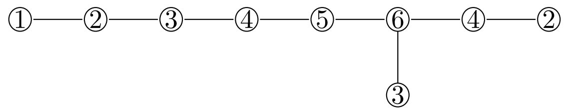
and hence we have at most one triple point, i.e. we cannot have more than three ends. If there is no triple point, our diagram is A_n, so assume not. The only remaining diagrams have one central vertex surrounded by three branches of length at least 1. The next diagram



tells us that we cannot have at least two edges in each direction, i.e. one direction has length exactly 1. We are aiming here for D_n or E_n so we eliminate the next case



which means that only one direction can have at least three edges, i.e. one direction has length 1, and another has length 1 or 2. We now only require the following



to show that the longest E_n diagram is E_8.

The only remaining Dynkin diagrams are those in our list. ■

We have shown that all root systems are of the specified types. We have yet, however, to show that root systems of each type exist, We want to construct a simple Lie algebra for each Dynkin diagram.

Theorem X.13. *Given a Dynkin diagram, we get a Cartan matrix, and we can define a Lie algebra by the following presentation. The generators are* $e_i := e_{\alpha_i}$, $f_i := f_{\alpha_i}$, $h_i := h_{\alpha_i}$. *The relations are*

$$\begin{aligned}
[h_i, h_j] &= 0 && \forall i, j \\
[h_i, e_j] &= \langle \alpha_j, \alpha_i \rangle e_j && \forall i, j \\
[h_i, f_j] &= -\langle \alpha_j, \alpha_i \rangle f_j && \forall i, j \\
[e_i, f_i] &= h_i && \forall i \\
[e_i, f_j] &= 0 && \forall i \neq j \\
\mathrm{ad}(e_i)^{1-\langle \alpha_j, \alpha_i \rangle}(e_j) &= 0 && \forall i \neq j \\
\mathrm{ad}(f_i)^{1-\langle \alpha_j, \alpha_i \rangle}(f_j) &= 0 && \forall i \neq j
\end{aligned}$$

The Lie algebra defined by this presentation is a simple Lie algebra. The Dynkin diagram of the Lie algebra is the Dynkin diagram we started with because the subspace spanned by $h_1, \ldots, h_r$ *is a Cartan subalgebra.*

Summary

Theorem X.14. *If* L *is a complex semisimple Lie algebra and* Φ_1, Φ_2 *are two root systems for* L *then* $\Phi_1 \cong \Phi_2$.

Corollary X.15. *A semisimple Lie algebra is a direct sum of simple Lie algebras* $L \cong L_1 \oplus \cdots \oplus L_k$, *and this decomposition is unique up to ordering and isomorphism.*

Proof. Take a Dynkin diagram of L. This decomposes, uniquely up to ordering, into connected components. These components correspond to summands $\Phi = \Phi_1 \cup \cdots \cup \Phi_k$. Then $L \cong L_1 \oplus \cdots \oplus L_k$ where each L_i has root system Φ_i. A Cartan subalgebra of L is $h = \langle h_1 \rangle \oplus \cdots \oplus \langle h_k \rangle$ where each $\langle h_i \rangle$ is a Cartan subalgebra of L_i. ■